A combinatorial parameterization of nilpotent orbits, twisted induction, and duality, I

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1. Basic Apparatus

Let $\mathfrak{g}$ be a semisimple Lie over $\mathbb{C}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{h};\mathfrak{g})$ the roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Pi \subset \Delta$, a set of simple roots and Δ^+ the corresponding set of positive roots. Let G be the adjoint group of $\mathfrak{g}$, and let $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$ be the cone of nilpotent elements of $\mathfrak{g}$. As is well-known $\mathcal{N}$ is the (disjoint) union of a finite number of G-orbits. We shall write $G\backslash\mathcal{N}$ indicate the set of nilpotent G-orbits in $\mathfrak{g}$.

2. Parameterizations of $G\backslash\mathcal{N}$

2.1. **Weighted Dynkin Diagrams.** Let $\mathcal{O}$ be a nilpotent orbit in $G\backslash\mathcal{N}$ and let $x \in \mathcal{O}$ be a representative element. By a theorem of Jacobson and Morozov, x extends to a standard triple $\{x, h, y\} \in \mathfrak{g}$, such that

$$[x,y] = h \quad , \quad [h,x] = 2 \quad , \quad [h,y] = -2y$$

and, moreover, h can be chosen such that h lies in the fundamental dominant Weyl chamber

$$D_\Delta; \{h' \in \mathfrak{h} \mid Re\left(\alpha\left(h'\right)\right) \geq 0 \quad \forall\ \alpha \in \Pi \text{ and whenever } Re\left(\alpha\left(h'\right)\right) = 0\ Im\left(\alpha\left(h'\right) \geq 0\right)\}$$

Theorem 2.1 (Kostant, [5])**.** *Suppose $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. A nilpotent orbit $\mathcal{O}$ is completely determined by the values $[\alpha_1(h), \alpha_2(h), \ldots, \alpha_n(h)]$. In fact, the only possible values of $\alpha_i(h)$ are 0, 1, and 2.*

Thus, each nilpotent orbits corresponds to a certain labeling of the nodes of Dynkin diagram of $\mathfrak{g}$ by one of $\{0, 1, 2\}$. Such a labeled Dynkin diagram is called a *weighted Dynkin diagram* (or *WDD*).

Remarks.
(1) Actually, relatively few of the $3^{rnk(\mathfrak{g})}$ possible weighted diagrams correspond to nilpotent orbits. Moreover, there is not even a rule for determining *a priori* which of these $3^{rnk(\mathfrak{g})}$ possibilities are actually realized as the WDD of a nilpotent orbit. For that reason, is perhaps more accurate to think of WWD's as providing a unique labeling of nilpotent orbits rather than a parameterization of nilpotent orbits.

(2) Nevertheless, an orbit's WWD does provide some interesting information about the orbit. For example,

- The relative size of an orbit is indicated somewhat by the number of non-zero nodes. In particular, the principal nilpotent orbit (the largest nilpotent orbit) *always* corresponds to the WDD consisting of only 2's and the trivial nilpotent orbit **0** *always* corresponds to the WDD consisting of only 0's.
- Orbits that consist of only 0 and 2's enjoy many special properties (and are called *even* nilpotent orbits).
- Any orbit that has a 2 in its Dynkin diagram is an *induced* nilpotent orbit (about which we shall say more later).

2.2. **Partition-type classifications.** When $\mathfrak{g}$ is a classical Lie algebra (i.e. a Lie algebra of Cartan type $A_n \approx SL_{n+1}(\mathbb{C})$, $B_n \approx SO(2n+1,\mathbb{C})$, $C_n \approx Sp(2n,\mathbb{C})$, $D_n \approx SO(2n,\mathbb{C})$), the nilpotent orbits of $\mathfrak{g}$ can be parameterized by partitions.

A partition $\mathbf{p}$ of N is a list $[p_1, \ldots, p_k]$ of non-negative integers such that

$$N = p_1 + p_2 + \cdots + p_k \qquad , \qquad p_1 \geq p_2 \geq \cdots \geq p_{k-1} \geq p_k \geq 0$$

and where it is tacitly assumed that

$$[p_1, \ldots, p_{k-1}, p_k] = [p_1, \ldots, p_{k-1}] \qquad \text{if } p_k = 0$$

The *multiplicity* of a part p_i of a partition $\mathbf{p} = [p_1, \ldots, p_k]$ is the number of times that particular integer p_i occurs in the list $[p_1, \ldots, p_k]$. One sometimes indicates a partition of N as a product of the form

$(p_1)^{m_1}(p_2)^{m_2}\cdots(p_k)^{m_k}$ where

$$p_1 > p_2 > \cdots > p_k > 0 \quad \text{and} \quad N = \sum_{i=1}^{k} m_i p_i$$

- The nilpotent orbits of $\mathfrak{sl}_n$ are in a one-to-one correspondence with the set $\mathcal{P}(n+1)$ of partitions of $n+1$.
- The nilpotent orbits of $\mathfrak{so}_{2n+1}$ are in a one-to-one correspondence with the set $\mathcal{P}_1(2n+1)$ consisting of partitions $2n+1$ such that even parts only occur with even multiplicity.
- The nilpotent orbits of $\mathfrak{sp}_{2n}$ are in a one-to-one correspondence with the set $\mathcal{P}_{-1}(2n)$ consisting of partitions $2n$ such that odd parts only occur with even multiplicity.
- The nilpotent orbits of $\mathfrak{so}_{2n}$ are in a *nearly* one-to-one correspondence with the set $\mathcal{P}_1(2n)$ consisting of partitions $2n$ such that even parts only occur with even multiplicity. Partitions in $\mathcal{P}_1(2n)$ which consist *only* of even parts (necessarily each with even multiplicity) are called *very even* partitions. To each very even partition there corresponds *two* distinct nilpotent orbits.

For the orbits of D_n, it is customary to append an additional label (I or II) to the very even partitions in order to maintain a parameterization in terms of (occasionally labeled) partitions.

Remarks 2.2. The partition-type classification provides a very natural and efficient way for dealing with the nilpotent orbits of classical groups. Not only is the set of possible parameters easy to determine but many properties of the orbits can be determined directly from their specification in terms of partitions. For example,

- The closure relations of the orbits corresponds directly to the standard dominance ordering of their respective partitions.
- Given a partition $\mathbf{p} \in \mathcal{P}_G$ (one of $\mathcal{P}(n)$, $\mathcal{P}_1(2n+1)$, $\mathcal{P}_{-1}(2n)$, $\mathcal{P}_1(2n)$), it is relatively easy to write down a representative element $x \in \mathcal{O}_{\mathbf{p}}$.
- There are nice algorithms for figuring out the dimension of an orbit $\mathcal{O}_{\mathbf{p}}$, the Spaltenstein dual $(\mathcal{O}_{\mathbf{p}})^{\vee}$ of an orbit, the orbits (in larger classical groups) obtained from $\mathcal{O}_{\mathbf{p}}$ by parabolic induction, etc.

2.3. **Bala-Carter Classification.** The biggest drawback of partition-type classifications is that they only apply to classical Lie algebras. According to the general philosophy espoused by Harish-Chandra, such a circumstance clearly belies a deficiency in understanding. A "good parameterization" of nilpotent orbits should be applicable to any semisimple Lie algebra. In two seminal papers ([1], [2]) appearing 1976 Bala and Carter achieved such a parameterization. Before describing the Bala-Carter classification we need to make a brief digression concerning two basic methods of "lifting" a nilpotent orbit in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ to a nilpotent orbit in $\mathfrak{g}$.

Definition 2.3. *Suppose $\mathcal{O}_{\mathfrak{l}}$ is a nilpotent orbit in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, the* Bala-Carter inclusion *of $\mathcal{O}_{\mathfrak{l}}$ is the nilpotent orbit obtained by applying G to the canonical embedding of $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{g}$.*

$$inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = G \cdot \mathcal{O}_{\mathfrak{l}}$$

(This you can think of as simply extending to L-orbit of an element $x \in \mathcal{O}_L$ to a G-orbit by regarding x as an element of $\mathfrak{g}$ and then acting by G.)

Definition 2.4. *Suppose $\mathcal{O}_{\mathfrak{l}}$ is a nilpotent orbit in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The G-orbit* induced from *$\mathcal{O}_{\mathfrak{l}}$ is*

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \textit{unique dense orbit in } G \cdot (\mathcal{O}_{\mathfrak{l}} + \mathfrak{u})$$

where $\mathfrak{u}$ is the nilradical of any *parabolic subalgebra $\mathfrak{p}$ which has $\mathfrak{l}$ as its Levi factor. (It is a theorem due to Luzstig and Spaltenstein that the G-orbit so constructed is independent of the choice of $\mathfrak{u}$.)*

We shall see later that, by a deep theorem of Barbasch and Vogan, these two constructions are actually dual to one another in a very precise sense.

By the way, there is special case of an induced orbit that we shall see quite prominently in what follows. We may as well define it here.

Definition 2.5. *A* Richardson orbit *is a nilpotent orbit that is induced from the trivial orbit* $\mathbf{0}$ *of a Levi subalgebra.*

The key idea in the Bala-Carter parameterization is the notion of a *distinguished nilpotent element.* This is just a nilpotent element that is not contained any proper Levi subalgebra. It was by studying the special properties of the orbits of distinguished elements in Levi subalgebras that Bala and Carter arrived their parameterization of the nilpotent orbits in $\mathfrak{g}$. However, in the end, the Bala-Carter parameterization is most easily described in terms of *distinguished parabolic subalgebras.* These are defined as follows.

Definition 2.6. *A parabolic subalgebra* $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ *of* $\mathfrak{g}$ *is called* distinguished *if*

$$\dim \mathfrak{l} = \dim \left(\mathfrak{u}/\left[\mathfrak{u},\mathfrak{u}\right]\right) \quad .$$

Theorem 2.7 (Bala-Carter, [1], [2])**.** *There is a natural one-to-one correspondence between nilpotent orbits of* $\mathfrak{g}$ *and* G*-conjugacy classes of pairs* $(\mathfrak{l},\mathfrak{p}_{\mathfrak{l}})$ *where* $\mathfrak{l}$ *is a Levi subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}_{\mathfrak{l}} = \mathfrak{m}+\mathfrak{u}$ *is a distinguished parabolic subalgebra of* $\mathfrak{l}$. *The correspondence is given by*

$$(\mathfrak{l},\mathfrak{p}_{\mathfrak{l}}) \to inc_{\mathfrak{l}}^{\mathfrak{g}}\left(ind_{\mathfrak{m}}^{\mathfrak{l}}\left(\mathbf{0}\right)\right) \tag{1}$$

Of course, at this point, this is not a very explicit parameterization, relying as it does on a classification of the conjugacy classes of pairs $(\mathfrak{l},\mathfrak{p}_{\mathfrak{l}})$. However, Bala-Carter had at their disposal Dynkin's 1957 classification ([4]) of conjugacy classes Levi subalgebras of semisimple Lie algebras, and it was fairly easy to identify the possible distinguished parabolic subalgebras of such a Levi.[1]

Now it turns out, in Dynkin's classification, that there is nearly a one-to-one correspondence between conjugacy classes of Levi subalgebras and the Cartan types of their semisimple part. In other words, it almost always happens that if $\mathfrak{l},\mathfrak{l}'$ are two Levi subalgebas of $\mathfrak{g}$ and $[\mathfrak{l},\mathfrak{l}]$ is isomorphic to $[\mathfrak{l}',\mathfrak{l}']$ as a semisimple Lie algebra, then $\mathfrak{l}$ and $\mathfrak{l}'$ are conjugate in $\mathfrak{g}$. For this reason, you can nearly always get away with labeling a conjugacy class of Levi subalgebras by the Cartan type of its semisimple part. For simple Lie algebras the exceptions to this simple circumstance are easily accounted for, and handled by adding some additional ornamentation to Cartan types. For example, F_4 has two conjugacy classes of Levis whose semisimple parts are of type A_1. The A_1 factor of one of this corresponds to an $\mathfrak{sl}_2$ subalgebra generated by a long root and the other to an $\mathfrak{sl}_2$ subalgebra generated by a short. These are two conjugacy classes are distinguised by writing A_1 for the "long" $\mathfrak{sl}_2$ and $\widetilde{A}_1$ for the "short" $\mathfrak{sl}_2$.

2.4. **Combinatorial Bala-Carter parameters.** When $\mathfrak{g}$ is not simple, or when actually wants to identify a set of generators of a Levi subalgebra, Dynkin's lists in terms of annotated Cartan types becomes a major pain. Bala and Carter's notation for the distinguished orbits is also only mildly informative even when it's deciphered.[2] Below we give a parameterization of nilpotent orbits in terms of certain pairs (Γ,γ) where Γ is a subset of the simple roots of $\mathfrak{g}$ and γ is a subset of Γ.

2.4.1. *Standard* Γ*'s.* Above Π denoted a choice of simple roots of $\mathfrak{g}$. Actually, in what follows, we shall think of Π more abstractly: first as the nodes of the Dynkin diagram of $\mathfrak{g}$, or as a list of integers $\{1,\ldots,rnk\left(\mathfrak{g}\right)\}$ corresponding to a labeling of the nodes of the Dynkin diagram of $\mathfrak{g}$ following Bourbaki conventions.

One standard way to producing a Levi subalgebra is to select a subset of Γ of Π and set

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\lambda\in span\langle\alpha_i\,|\,i\in\Gamma\rangle} \mathfrak{g}_\lambda$$

The correspondence

$$\Pi \supset \Gamma \to \{\text{conjugacy class of } \mathfrak{l}_\Gamma\}$$

[1]Spoiler Alert: Note that as the notation on the right hand side of (1) suggests the Bala-Carter correspondence between nilpotent orbits and G-conjugacy classes of distinguished parabolic subalgebras of levi subalgebras is reducible to a correspondence between nilpotent orbits and G-conjugacy classes of certain "flags" of Levi subalgebras $(\mathfrak{l},\mathfrak{m})$, $\mathfrak{m}\subseteq\mathfrak{l}$. In fact, this correspondence will be reduced further to a correspondence between W-conjugacy classes of pairs (Γ,γ) where Γ is a subset of the simple roots of $\mathfrak{g}$ and γ is a "distinguished" subset of Γ .

[2]E.g. the distinguished orbits of E_8 are denoted by E_8, $E_8\left(a_1\right)$, $E_8\left(a_2\right)$, $E_8\left(a_3\right)$, $E_8\left(a_4\right)$, $E_8\left(b_4\right)$, $E_8\left(a_5\right)$, $E_8\left(b_5\right)$, $E_8\left(a_6\right)$, $E_8\left(b_6\right)$, and $E_8\left(a_7\right)$.

is actually surjective onto the set of conjugacy classes of Levi subalgebras, but it is also many-to-one in general. However, there is a simple criterion that decides when two such Levi subalgebras lie in the same conjugacy class:

Theorem 2.8 (Dynkin, [4])**.** *Two Levi subalgebras $\mathfrak{l}_\Gamma$ and $\mathfrak{l}_{\Gamma'}$ are conjugate in $\mathfrak{g}$ if and only if Γ is conjugate to Γ' under the Weyl group of $\mathfrak{g}$.*

Thus, an easy way to parameterize the conjugacy classes of Levi subalgebras is to

- construct the power set 2^Π (the set of all subsets of Π)
- partition 2^Π into W-conjugacy classes $\widetilde{\Gamma}$
- select from each such W-conjugacy class a representative Γ.

A collection $\Psi = \{\Gamma_1, \ldots, \Gamma_k\}$ so obtained will be called a collection of *standard* Γ*'s*. It follows from Dynkin's theorem that the correspondence

$$\Psi \to \{\text{conjugacy classes of Levi subalgebras}\} \quad : \quad \Psi \ni \Gamma \to \mathfrak{l}_\Gamma \in G \cdot l_\Gamma$$

is one-to-one.

Here is an even more expedient way of generating a collection of standard Γ's using John Stembridge's Coxeter package (and Maple).

```
MkStandardGammas := proc(Gtype)
local i,r,PS,SGs,p,cr:
r := rank(Gtype):
PS := map(x->convert(x,list),combinat[powerset](r)):
SGs := {}:
for i from 1 to nops(PS) do
   p := op(i,PS):
   cr := coxeter[class_rep](p,Gtype):
   SGs := SGs union {p}:
od:
# sort SGs by cardinality
SGs := sort(SGs,'length'):
end:
```

2.4.2. *Distinguished* γ*'s.* Now we turn out attention to parameterizing the distinguished parabolics of a semisimple Lie algebra. Since we are most interested in the case when the semisimple Lie algebra is the semisimple part of a Levi subalgebra $\mathfrak{l}$ of some other Lie algebra, we may as well use $\mathfrak{l}$ to indicate the semisimple algebra in which the distinguished parabolics live.

Fact 2.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and simple root system Π. For any subset $\gamma = \{i_1, \ldots, i_k\} \subset \Pi$, let*

$$\Delta_c = \{\lambda \in span_{\mathbb{Z}}(\alpha_{i_1}, \ldots, \alpha_{i_k})\}$$

and set $\mathfrak{p}_\gamma = \mathfrak{l}_\gamma + \mathfrak{u}_\gamma$ where

$$\mathfrak{l}_\gamma = \mathfrak{h} + \sum_{\alpha \in \Delta_\gamma} \mathfrak{g}_\alpha$$

$$\mathfrak{u}_\gamma = \sum_{\alpha \in \Delta^+ - \Delta_\gamma} \mathfrak{g}_\alpha$$

Then the correspondence

$$2^\Pi \to \{\textit{conjugacy classes of parabolic subalgebras}\} \quad ; \quad 2^\Pi \ni \gamma \to G \cdot \mathfrak{p}_\gamma$$

is a bijection.

We thus have an extremely simple combinatorial parameterization of conjugacy classes of parabolic subalgebras. What we need is a simple test to figure out which of these parabolic subalgebras is distinguished.

To do this we first note that every root α in $\Delta^+ - \Delta_\gamma$ has at least one simple root component outside of γ and because of this is α is the root corresponding to a root vector in $[\mathfrak{u}_\gamma, \mathfrak{u}_\gamma]$ if and only if α has at least two simple root components outside of γ. Since each root space is 1-dimensional

$$\dim\left(\mathfrak{u}_\gamma / \left[\mathfrak{u}_\gamma, \mathfrak{u}_\gamma\right]\right) = \#\left\{\text{roots in } \Delta^+ - \Delta_c \text{ with exactly one simple root component outside of } \gamma\right\}$$

On the other hand,

$$\dim\left(\mathfrak{l}_\gamma\right) = rnk\left(\mathfrak{g}\right) + \left|\Delta_\gamma\right|$$

So if we set

$$n_1\left(\gamma, \Pi\right) \equiv rnk\left(\mathfrak{g}\right) + \left|\Delta_\gamma\right|$$
$$n_2\left(\gamma, \Pi\right) \equiv \#\left\{\text{roots in } \Delta^+ - \Delta_c \text{ with exactly one simple root component outside of } \gamma\right\}$$

the a parabolic subalgebra $\mathfrak{p}_\gamma$ will be distinguished in $\mathfrak{g}$ if and only if $n_1(\gamma) = n_2(\gamma)$.

The construction of the set $\mathcal{BC}$ of *combinatorial Bala-Carter parameters*. One first constructs a set Ψ of standard Γ's. $\Gamma \in \Psi$ will be in a one-to-one correspondence with G-conjugacy classes of Levi subalgebras via the correspondence

$$\Gamma \to \mathfrak{l}_\Gamma$$

Next, for each $\Gamma \in \Psi$ we run through the power set of Γ and look for distinguished γ inside Γ, that is to say, we identify the subsets γ of Γ for which

$$n_1\left(\gamma, \Gamma\right) = n_2\left(\gamma, \Gamma\right) \tag{1}$$

The pairs (Γ, γ) so obtained will constitute the set $\mathcal{BC}$. We remarked that the condition (1) is also easily checked using John Stembridge's Coxeter package.

```
DistinguishedTest := proc(gamma,Gamma,Gt)
local i,SR,LB,lb,gc,PR,pr,n1,n2,x,q:
if gamma = [] then RETURN(1): fi:
SR := coxeter[base](Gt):
LB := [seq(SR[op(i,Gamma)],i=1..nops(Gamma))]:
lb := [seq(SR[op(i,gamma)],i=1..nops(gamma))]:
gc := []:
for i from 1 to nops(LB) do
  if not member(op(i,Gamma),convert(gamma,set)) then
    gc := [op(gc),i]:
  fi:
od:
PR := coxeter[pos_roots](LB):
pr := coxeter[pos_roots](lb):
n1 := 2*nops(pr) + Rank(LB):
n2 := 0:
for i from 1 to nops(PR) do
  x := coxeter[root_coords](op(i,PR),LB):
  q := add(x[op(j,gc)],j=1..nops(gc)):
  if q = 1 then
    n2 := n2+1:
  fi:
od:
if n1 = n2 then
  RETURN(1):
else
  RETURN(0)
fi:
```

```
end:
```

Obviously, via this construction we have

Theorem 2.10. *The pairs* $(\Gamma,\gamma)\in\mathcal{BC}$ *are in one-to-one correspondence with* G*-conjugacy classes of pairs* $(\mathfrak{l},\mathfrak{p}_{\mathfrak{l}})$ *where* l *is a Levi subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}_{\mathfrak{l}}$ *is a distinguished parabolic subalgebra of* $\mathfrak{l}$*. Moreover, the pairs* $(\Gamma,\gamma)\in\mathcal{BC}$ *parameterize the nilpotent orbits of* $\mathfrak{g}$*. The correspondence is given by*

$$(\Gamma,\gamma)\to inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}(\mathbf{0})\right)$$

2.5. **CBC diagrams.** A modified Dynkin diagram provides a succinct way of specifying the combinatorial Bala-Carter parameters (Γ,γ) of nilpotent orbit $\mathcal{O}_{[\Gamma,\gamma]}$ in $\mathfrak{g}$. One starts with the Dynkin diagram for $\mathfrak{g}$ using open circles at each node. The nodes corresponding to the indices in γ are then denoted by asterixs and the remaining nodes in Γ are shaded black. We call these gadgets *BC diagrams.* In Appendix A we give the CBC diagrams of the nilpotent orbits of the exceptional Lie algebras. Below we'll give simple recipes for going back and forth between CBC diagrams and the partition parametization of nilpotent orbits of the classical Lie algebras.

2.6. **Connection with Partition Classification.** As remarked above the partition classification scheme is an extremely useful way of parameterizing the nilpotent orbits of a classical Lie algebra. It turns out though that it is quite easy to go from our combinatorial Bala-Carter parameters to the corresponding partition. So that the reader will bear with us for a moment, we should point out that while it takes a bit of work to develop these recipes, they will be quite trivial to implement in practice.

Recall the orbit $\mathcal{O}_{(\Gamma,\gamma)}$ is the inclusion (G-saturation) of a distinguished orbit $\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}$ inside a Levi subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$. The parameter Γ is a subset of the simple roots Π that generate the semisimple part of the Levi subalgebar $\mathfrak{l}_\Gamma$ and γ is a "distinguished" subset of Γ specifying a parabolic subalgebra of $\mathfrak{l}_\Gamma$ such that

$$\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}=ind_{\mathfrak{p}_\gamma}^{\mathfrak{l}_\Gamma}(\mathbf{0})$$

is a distinguished nilpotent orbit in $\mathfrak{l}_\Gamma$.

Now, on the level of representative elements, the *inclusion* of the distinguished orbit $\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}$in $\mathfrak{l}_\Gamma$ into $\mathfrak{g}$ is quite literal. For the representative $x\in\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}\subset\mathcal{N}_{\mathfrak{l}}$ is immediately interpretable as a representative element of $\mathcal{O}_{(\Gamma,\gamma)}\subset\mathcal{N}_{\mathfrak{g}}$. In terms of the defining matrix realization of $\mathfrak{g}$, the Levi subalgebra $\mathfrak{l}_\Gamma$ will correspond to a certain diagonal block and a distinguished element of $\mathfrak{l}_\Gamma$ will simply look like

$$\left[\begin{array}{ccc}\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&x&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\end{array}\right]\quad,\quad x\text{ a distinguished element in the block corresponding to }\mathfrak{l}_\Gamma$$

Now the partition classification parameterizes nilpotent orbits by the Jordan canonical form of representative elements. It is thus obvious that the partition $\mathbf{p}_{(\Gamma,\gamma)}$ corresponding to the orbit $\mathcal{O}_{(\Gamma,\gamma)}$ is should be the partition corresponding to the orbit of $x\in\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}\subset\mathfrak{l}_\Gamma$ with some 1's added. The added 1's just correspond to the trivial Jordan blocks in $\mathfrak{g}$ lying outside the $\mathfrak{l}_\Gamma$ block in $\mathfrak{g}$.

The first case to consider is when $\Gamma=\Pi$; that is to say, when $\mathfrak{l}_\Gamma=\mathfrak{g}$, and $\mathcal{O}_{(\Gamma,\gamma)}$ is simply a distinguished orbit of $\mathfrak{g}$. For this we simply quote a result proved in Collingwood-McGovern.

Fact 2.11 (pg. 126 of [3])**.** *The partitions corresponding to distinguished orbits in classical groups are as follows:*

- $\mathfrak{sl}_n$ *There is only one distinguished orbit in* $\mathfrak{sl}_n$*. It is the principal orbit and the corresponding partition is* $[n]$.
- $\mathfrak{so}(2n+1)$ *The distinguished orbits of* $\mathfrak{so}(2n+1)$ *correspond to strictly decreasing partitions* $\mathbf{p}=[p_1,\ldots,p_k]$ *of* $2n+1$ *consisting of only odd parts.*
- $\mathfrak{sp}(2n)$ *The distinguished orbits of* $\mathfrak{sp}(2n)$ *correspond to strictly decreasing partitions* $\mathbf{p}=[p_1,\ldots,p_k]$ *of* $2n$ *consisting of only even parts*
- $\mathfrak{so}(2n)$ *The distinguished orbits of* $\mathfrak{so}(2n)$ *correspond to strictly decreasing partitions* $\mathbf{p}=[p_1,\ldots,p_k]$ *of* $2n$ *consisting of only odd parts*

We'll now connect these characterizations of "distinguished partitions" with distinguished subsets γ of Π. By definition, the distinguished orbit correponding to (Π, γ) in $\mathfrak{g}$ is the Richardson orbit of the standard parabolic $\mathfrak{p}_\gamma$ corresponding to the subset γ of the simple roots Π of $\mathfrak{g}$.

When $\gamma = \{\}$ the parabolic $\mathfrak{p}_{\{\}} = \mathfrak{t} + \mathfrak{n}$ is the standard Borel subalgebra of $\mathfrak{g}$ whose corresponding Richardson orbit is the principal orbit. Below we list the partitions $\mathbf{p}_{prin}$ corresponding to the principal orbits of classical Lie algebras.

$\mathfrak{g}$	$\mathbf{p}_{prin}$
$\mathfrak{sl}_n$	$[n]$
$\mathfrak{so}_{2n+1}$	$[2n+1]$
$\mathfrak{sp}_{2n}$	$[2n]$
$\mathfrak{so}(2n)$	$[2n-1, 1]$

For the remaining cases, when $\gamma \neq \{\}$ we can assume that $\mathfrak{g} \neq \mathfrak{sl}_n$, because $\mathfrak{sl}_n$ has only one distinguished orbit, the principal orbit considered above.

In the midst of my December seminars I developed a recipe for the figuring out the partition corresponding to the Richardson orbit

$$ind_{\mathfrak{l}_\gamma}^{\mathfrak{g}}(\mathbf{0})$$

of a classical group. Rather than repeat that discussion, I'll summarize the recipe, restricted to the case at hand.

Proposition 2.12. *Let Π be the simple roots of a classical Lie algebra of type $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(2n)$, or $\mathfrak{so}(2n)$ ordered via Bourbaki conventions. Let $\gamma \subset \{1, 2, \ldots, n\}$ be the indices of a distinguished subset simple roots, regarded as an ordered list of integers, and let $\mathfrak{p}_\gamma = \mathfrak{l}_\gamma + \mathfrak{u}_\gamma$ be the corresponding standard parabolic subalgebra of $\mathfrak{g}$. The partition of $\mathcal{P}_G$ corresponding to a distinguished Richardson orbit*

$$ind_{\mathfrak{l}_\gamma}^{\mathfrak{g}}(\mathbf{0})$$

can be obtained from γ as follows.

- *Split γ into its G-tail γ_G and A-head γ_A; where*
$$\gamma_G = \text{ maximal strictly consecutive subsequence of } \gamma \text{ terminating with } n$$
$$\gamma_A = \text{subsequence of } \gamma \text{ obtained by chopping off the } G\text{-tail from } \gamma$$
- *Let r be the length of γ_G. Let $\mathbf{d}_G$ be the partition corresponding to the trivial representation of the classical Lie algebra $\mathfrak{g}_r$ of the same Cartan type as $\mathfrak{g}$ but of rank r. This will just be $\left[(1)^{d_r}\right]$ where d_r is the dimension of the standard representation of $\mathfrak{g}_r$. If $r = 0$ and $\mathfrak{g}$ is of type $\mathfrak{so}(2n+1)$ take $\mathbf{d}_G = [1]$, otherwise if $r = 0$ take $\mathbf{d}_G = []$.*
- *Let $m_1, \ldots, m_k$ be the lengths of the maximal strictly consecutive subsequences of γ_A and let*
$$\mathbf{m}_A = [m_1 + 1, m_2 + 1, \ldots, m_k + 1]$$
and set
$$\mathbf{d}_A = (\mathbf{m}_A)^t$$
- *If necessary, extend either $\mathbf{d}_A$ or $\mathbf{d}_G$ with 0's so that both $\mathbf{d}_A$ and $\mathbf{d}_G$ have the same length. Let k denote this common length.*
- *Form the partition $\mathbf{p}_\gamma = [p_1, \ldots, p_k]$ by setting*
$$p_i = 2(\mathbf{d}_A)_i + (\mathbf{d}_G)_i$$
This will automatically be a partition in $\mathcal{P}(d_n)$, however, it need not satisfy the parity conditions of $\mathcal{P}_G$.
- *The partition corresponding to the Richardson orbit $ind_{\mathfrak{l}_\gamma}^{\mathfrak{g}}(\mathbf{0})$ will be the partition G-collapse $(\mathbf{p}_\gamma)_G$ of $\mathbf{p}_\gamma$. $(\mathbf{p}_\gamma)_G$ is by definition the (unique) maximal (w.r.t. to dominance ordering of partitions) partition in $\mathcal{P}_G$ that is dominated by $\mathbf{p}_\gamma$.*

Remark 2.13. To get the partition corresponding to an arbitrary Richardson orbit (as opposed to a distinguished Richardson orbit) one just needs to extend the above recipe by one more step; taking the corresponding G-collapse of $\mathbf{p}_\gamma$. In restricting to distinguished Richardson orbits, we also avoided the labeling issues associated with the even-even Richardson orbits of $\mathfrak{so}(2n)$.

In Appendix B we provide a table listing the CBC diagrams (from which the γ's can be read) and the partitions of the distiguished orbits of each classical Lie algebra of rank ≤ 8.

Next, we consider the cases when Γ is a proper subset of Π (and $\mathfrak{l}_\Gamma$ is proper Levi subalgebra of $\mathfrak{g}$) and we'll proceed case-by-case for each Cartan type.

2.6.1. $\mathfrak{sl}_n$. The CBCPs for $\mathfrak{sl}_n$ are always of the form $[\Gamma, []]$; since all the Levis of $\mathfrak{sl}_n$ are direct sums of $\mathfrak{gl}_k$'s and the only distinguished orbit in a $\mathfrak{gl}_k$ is the principal orbit. Let $\mathbf{m}_\Gamma = [m_1, \ldots, m_k]$ be the lengths of the strictly consecutive subsequences in Γ: For example, if

$$\Gamma = [1, 2, 3, 5, 6, 8, 10]$$

then

$$\mathbf{m}_\Gamma = [3, 2, 1, 1]$$

The partition corresponding to $\mathcal{O}_{[\Gamma,[]]}$ is obtained by adding 1 to each entry in $\mathbf{m}_\Gamma$ and then adjoining sufficiently many 1's at the end so that the sum of the entries is $n+1$. In this way, one obtains a partition $\mathbf{p}_\Gamma$ of $n+1$. The partition corresponding to the orbit $\mathcal{O}_{[\Gamma,[]]}$ is the partition $\mathbf{p}_\Gamma$. This works because a maximal strictly consecutive subsequence of length m in Γ corresponds to an $\mathfrak{sl}_{m+1}$ summand in $\mathfrak{l}_\Gamma$. Thus,

$$m_\Gamma = [m_1, \ldots, m_k] \quad \Longrightarrow \quad \mathfrak{l}_\Gamma = \mathfrak{gl}_{m_1+1} \oplus \mathfrak{gl}_{m_2+1} \oplus \cdots \oplus \mathfrak{gl}_{m_k+1}$$

Embedding the principle orbits (the only distinguished orbits) for these $\mathfrak{gl}_{m_i+1}$blocks into $\mathfrak{sl}_n$ will produce a element of $\mathfrak{sl}_n$ whose Jordan form consists of Jordan blocks of size m_1+1, $m_2+1, \ldots, m_k+1$ and however many (trivial) Jordan blocks of size 1 it takes the fill the rest of the representative matrix in $\mathfrak{sl}_n$ with 0's.

We remark that this is even simpler than the recipe (due to Kraft, Ozeki and Wakimoto) for finding the partition for a Richardson orbit in SL_{n+1} corresponding to the Levi subalgebra of $\mathfrak{sl}_{n+1}$ specified by Γ. (See, e.g., page 112 of [3]. For we don't need to utilize the partition transpose map. The reason for the simplification is that we are including principal orbits of Levi subalgebras rather than inducing trivial orbits of Levi subalgebras (as in the Kraft, Ozeki, Wakimoto situation). Of course, the operation of including principal orbits of a Levi is naturally dual to the operation of inducing its trivial orbit (in the framework of Barbasch-Vogan). And in the partition classification of nilpotent orbits for A_n, where every orbit is both a "principal include" and a Richardon orbit, this duality between "principal includes" and Richardson orbits can be implemented via the transpose map.

The other classical groups are not much harder; however, the recipes require a wee bit of preparation.

Suppose $\mathfrak{g}$ is of type B_n, C_n or D_n. Then any Levi subalgebra of $\mathfrak{g}$ is of the form[3]

$$\mathfrak{l} = \mathfrak{gl}_{i_1} + \mathfrak{gl}_{i_2} + \cdots + \mathfrak{gl}_{i_k} + \mathfrak{g}_r$$

where $\mathfrak{g}_r$ is of the same Cartan type as $\mathfrak{g}$ but of rank r, and one has

$$d_r + 2i_1 + \cdots + 2i_k = d_n$$

Here d_n denotes the dimension of the standard representation of $\mathfrak{g}$ (when $rnk\,(\mathfrak{g}) = n$).

To see how this comes about suppose that Γ is given as a ordered list of integers between 1 and $n = rnk\,(\mathfrak{g})$, and that moreover suppose we are following Bourbaki conventions so that characteristic part of the Dynkin diagram; the short simple root for $\mathfrak{so}\,(2n+1)$, the long simple root for $\mathfrak{sp}\,(2n)$ or the final split pair of simple roots for $\mathfrak{so}\,(2n)$ correspond to the last simple root, or last pair of simple roots for $\mathfrak{so}\,(2n)$. If r is the length of the longest strictly consecutive subsequence in Γ that terminates on n (r will be 0 if Γ does not terminate with n), then there will be a summand of $\mathfrak{l}_\Gamma$ of the same Cartan type as $\mathfrak{g}$ but of rank r. We call that subsequence

$$[n-r, n-r+1, \cdots, n] \subset \Gamma$$

the G-tail Γ_G of Γ. The A-head Γ_A of Γ is will correspond to a series of $\mathfrak{gl}_k$ summands of $\mathfrak{l}_\Gamma$. Just as in the case of $\mathfrak{sl}_n$ above, the ranks of these $\mathfrak{gl}_k$ summands can be obtained by figuring out the lengths of the maximal strictly consecutive subsequences of Γ_A, and increasing each of these by 1.

[3]If not already apparent, this will become so in the next paragraph.

For example, if $\mathfrak{g} = \mathfrak{so}(9)$ and $\Gamma = [1,2,3,6,8,9]$, we have

$$\Gamma_G = [8,9]$$
$$\Gamma_A = [1,2,3,6]$$

and

$$\mathfrak{l}_\Gamma \approx \mathfrak{g}l_4 \oplus \mathfrak{gl}_2 \oplus \mathfrak{so}(5)$$

The number r is just the cardinality of Γ_G and the numbers $i_1, \ldots, i_k$ are just the lengths plus one of the strictly consecutive subsequences of Γ_A. (There is some additional fussing around for the case of D_n but we'll get to that in a second.)

Now recall that the nilpotent orbit $\mathcal{O}_{(\Gamma,\gamma)}$ is simply the *inclusion* of the distinguished orbit $\mathcal{O}_{\mathfrak{l}_{\Gamma,\gamma}}$ in $\mathfrak{l}_\Gamma$ corresponding to the distinguished subset $\gamma \subset \Gamma$. Since this *inclusion* is quite literal, a representative $x \in \mathcal{O}_{\mathfrak{l}_{\Gamma,\gamma}} \subset \mathcal{N}_{\mathfrak{l}}$ is immediately interpretable as a representative element of $\mathcal{O}_{(\Gamma,\gamma)} \subset \mathcal{N}_{\mathfrak{g}}$. E.g., in terms of the defining matrix realization of $\mathfrak{g}$, the Levi subalgebra $\mathfrak{l}_\Gamma$ will correspond to a certain diagonal block and and are representative nilpotent x of $\mathcal{O}_{(\Gamma,\gamma)}$ will be of the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\widetilde{x}$ is a nilpotent matrix in the defining matrix realization of the (isomorphism class of the) semisimple part of $\mathfrak{l}_\Gamma$. Now the partition classification parameterizes nilpotent orbits by the Jordan canonical form of representative elements. It is thus obvious that the partition $\mathbf{p}_{(\Gamma,\gamma)}$ corresponding to the orbit $\mathcal{O}_{(\Gamma,\gamma)}$ is just the partition corresponding to the orbit of $\widetilde{x} \in \mathcal{O}_{\mathfrak{l}_{\Gamma,\gamma}}$ with some 1's added corresponding to the trivial Jordan blocks lying outside of $\mathfrak{l}_\Gamma$ in the natural embedding $\mathfrak{l}_\Gamma$. In other words, the partition corresponding to $\mathcal{O}_{(\Gamma,\gamma)} \equiv inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}\right)$ is just the partition corresponding to $\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma} \subset \mathfrak{l}_\Gamma$ padded with sufficiently many 1's to get a partition of d_G. (It will turn out that partition so obtained will automatically satisfy the parity condition of $\mathcal{P}_G$.)

We'll now describe how to determine the partition $\mathbf{p}_{\Gamma,\gamma}$ corresponding to a distinguished orbit $\mathcal{O}_{\mathfrak{l}_\Gamma,\gamma}$ in the Levi $\mathfrak{l}_\Gamma$. Recall that by breaking Γ up into its G-tail and A-head, we can realize $\mathfrak{l}_\Gamma$ as

$$\mathfrak{l}_\Gamma = \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_k} \oplus \mathfrak{g}_r$$

Accordingly, a representative element x of a distinguished nilpotent orbit in $\mathfrak{l}_\Gamma$ will split across the various summands of $\mathfrak{l}_\Gamma$. In fact, writing

$$x = x_{i_1} + x_{i_k} + x_r \qquad , \quad x_{i_1} \in \mathfrak{gl}_{i_1}, \ \cdots , \ x_{i_k} \in \mathfrak{gl}_{i_k}, \ x_r \in \mathfrak{g}_r$$

each of the "components" $x_{i_1}, \ldots, x_r$ will be a distinguished element of the corresponding summand of $\mathfrak{l}_\Gamma$. Moreover, the Jordan form of x will simply be a concatenation of the Jordan blocks corresponding to $x_{i_1}, \ldots, x_r$. And so the partition $\mathbf{p}_{\Gamma,\gamma}$ will just be a concatenation of the partitions corresponding to $x_{i_1}, \ldots, x_r$.

Now if x_{i_j} is a distinguished nilpotent element of a $\mathfrak{gl}_{i_j}$ factor, then it must in fact correspond to a principal nilpotent element of $\mathfrak{sl}_{i_j}$ (for $\mathfrak{sl}_k$ the only distinguished orbits are principal orbits). Thus, x_{i_j} can be represented as a sum of the root vectors corresponding to the simple roots of $\mathfrak{sl}_{i_j}$. These simple root vectors will actually be strictly consecutive simple root vectors for $\mathfrak{g}$. Thus the contribution of x_{i_1} to the partition $\mathbf{p}_{\Gamma,\gamma}$ will be the partition corresponding to the Jordan form of a strictly consecutive sum of simple roots of $\mathfrak{g}$. Looking at any of the standard Chevalley bases for the defining representation of $\mathfrak{g}$ one sees that the Jordan form of a strictly consecutive sum of k simple roots (excluding the "characteristic simple root" α_n will not anyway appear in the A-head of Γ) of $\mathfrak{g}$ will consist of two identical Jordan blocks of size k. Thus, each component x_{i_j} in a $\mathfrak{gl}_{i_j}$ factor of $\mathfrak{l}_\Gamma$ will contribute a pair (i_j, i_j) to the partition $\mathbf{p}_{\Gamma,\gamma}$. This means that if the lengths of the maximal strictly consecutive indices in the A-head of Γ are $(m_1, \ldots, m_k)$ then the contribution of the A-head to $\mathbf{p}_{\Gamma,\gamma}$ will be

$$[m_1 + 1, m_1 + 1, \ldots, m_k + 1, m_k + 1]$$

Let us now turn our attention to the contribution of the G-tail Γ_Γ of Γ. Now it is actually in Γ_G that the distinguished subset $\gamma \in \Gamma$ resides. So we need to figure out the partition corresponding to a distinguished orbit (Γ_G, γ) in $\mathfrak{l}_{\Gamma_G}$. But this we have already done in Proposition 2.1.2.

We can now give the algorithm for attaching a partition in $\mathcal{P}_G$ to a combinatorial Bala-Carter parameter (Γ, γ) when G is of type $\mathfrak{so}\,(2n+1)$, $\mathfrak{sp}\,(2n)$. The case of $\mathfrak{so}\,(2n)$ has a minor complication related to the existence of even-even orbits. But we'll soon get to that soon enough.

Proposition 2.14. *Let (Γ, γ) be a combinatorial Bala-Carter parameter for nilpotent orbit of $\mathfrak{g} = \mathfrak{so}\,(2n+1)$ or $\mathfrak{sp}\,(2n)$. Then the partition $\mathbf{p}_{(\Gamma,\gamma)}$ corresponding corresponding to $\mathcal{O}_{(\Gamma,\gamma)} \subset \mathcal{N}_{\mathfrak{g}}$ is obtained as follows.*

- *Split Γ into its A-head Γ_A and G-tail Γ_G.*
- *Determine the lengths $m_1, \ldots, m_k$ of the strictly consecutive subsequences of occuring in Γ_A. Set*
$$\mathbf{m}_A = [m_1+1, m_1+1, m_2+1, m_2+1, \cdots, m_k+1, m_k+1]$$
- *Use the algorithm of Proposition 2.12 to determine the partition $\left(\mathbf{p}_{(\Gamma_G,\gamma)}\right)_G$ corresponding to the distinguished Richardson orbit $ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_{\Gamma_G}}(\mathbf{0})$.*
- *Concatenate $\mathbf{m}_A$ with $\left(\mathbf{p}_{(\Gamma_G,\gamma)}\right)_G$ and add as many 1's as needed to the result so as to obtain finally a partition of $\mathcal{P}_G$.*

Example 2.15. $\mathfrak{so}\,(9)$.

There are 13 possible combinatorial Bala-Carter parameters.

Diagram	$[\Gamma,\gamma]$	$\mathfrak{l}_\Gamma$	Γ_A	$\mathbf{m}_A$	Γ_G	$\left(\mathbf{p}_{(\Gamma_G,\gamma)}\right)_G$	$\mathbf{p}_{(\Gamma,\gamma)}$
$\circ-\circ-\circ\Longrightarrow\circ$	$[[],[]]$	$\mathbf{0}$	$[]$	$[]$	$[]$	$[]$	$\left[(1)^9\right]$
$\bullet-\circ-\circ\Longrightarrow\circ$	$[[1],[]]$	A_1	$[1]$	$[2,2]$	$[]$	$[]$	$\left[(2)^2,(1)^5\right]$
$\circ-\circ-\circ\Longrightarrow\bullet$	$[[4],[]]$	B_1	$[4]$	$[]$	$[4]$	$[3]$	$\left[3,(1)^6\right]$
$\bullet-\circ-\circ\Longrightarrow\bullet$	$[[1,4],[]]$	A_1+B_1	$[1]$	$[2,2]$	$[4]$	$[3]$	$\left[3,(2)^2,(1)^2\right]$
$\bullet-\circ-\bullet\Longrightarrow\circ$	$[[1,3],[]]$	$2A_1$	$[1,3]$	$[2,2,2,2]$	$[]$	$[]$	$\left[(2)^4,1\right]$
$\circ-\circ-\bullet\Longrightarrow\bullet$	$[[3,4],[]]$	B_2	$[]$	$[]$	$[3,4]$	$[5]$	$\left[5,(1)^4\right]$
$\bullet-\bullet-\circ\Longrightarrow\circ$	$[[1,2],[]]$	A_2	$[1,2]$	$[3,3]$	$[]$	$[]$	$\left[(3)^2,(1)^3\right]$
$\bullet-\bullet-\circ\Longrightarrow\bullet$	$[[1,2,4],[]]$	A_2+B_1	$[1,2]$	$[3,3]$	$[4]$	$[3]$	$\left[(3)^3\right]$
$\bullet-\circ-\bullet\Longrightarrow\bullet$	$[[1,3,4],[]]$	A_1+B_2	$[1]$	$[2,2]$	$[3,4]$	$[5]$	$\left[5,(2)^2\right]$
$\circ-\bullet-\bullet\Longrightarrow\bullet$	$[[2,3,4],[]]$	B_3	$[]$	$[]$	$[2,3,4]$	$[7]$	$\left[7,(1)^2\right]$
$\bullet-\bullet-\bullet\Longrightarrow\circ$	$[[1,2,3],[]]$	A_3	$[1,2,3]$	$[4,4]$	$[]$	$[]$	$\left[(4)^2,1\right]$
$\bullet-*-\bullet\Longrightarrow*$	$[[1,2,3,4],[2,4]]$	$B_4\,(a_2)$	$[]$	$[]$	$[1,2,3,4]$	$[5,3,1]$	$[5,3,1]$
$\bullet-\bullet-\bullet\Longrightarrow\bullet$	$[[1,2,3,4],[]]$	B_4	$[]$	$[]$	$[1,2,3,4]$	$[9]$	$[9]$

We now turn our attention to the case of $\mathfrak{so}\,(2n)$. What's peculiar for this case is that there are pairs of non-conjugate Levi subalgebras $\mathfrak{l}_\Gamma$, $\mathfrak{l}_{\Gamma'}$ that are related by an outer automorphism of $\mathfrak{g}$ arising from the automorphism $\alpha_{n-1} \longleftrightarrow \alpha_n$ of the Dynkin diagram. The pairs Γ, Γ' for which this occurs can be characterized as follows: for one of the pair, say, Γ, the penultimate simple root α_{n-1} appears but the last simple root α_n does not; the other member of the Γ' is identical to Γ except that the last root α_n appears instead of α_{n-1}. The corresponding Levi subalgebras $\mathfrak{l}_\Gamma$ and $\mathfrak{l}_{\Gamma'}$ will both be isomorphic to a direct sum of $\mathfrak{gl}_{i_i}$, the ranks of which being determined by the lengths of the maximal subsequences of strictly consecutive subsequences.

Here is how we handle $\mathfrak{so}\,(2n)$. If Γ is not of the form of one these special pairs (i.e. if Γ involves one of and only one of the pair $\{\alpha_{n-1}, \alpha_n\}$) then we precede exactly as we did with the cases of $\mathfrak{so}\,(2n+1)$ and $\mathfrak{sp}\,(2n)$.

If Γ involves α_{n-1} but not α_n, the corresponding Levi will be isomorphic to a direct sum of $\mathfrak{gl}_k$'s withe the ranks k corresponding to the lengths (plus one) of the maximal strictly consecutive subsequences of Γ. Following the same prescription as above we get

$$\mathbf{m}_A = [m_1+1, m_1+1, \cdots, m_k+1, m_k+1] \quad , \quad \left(\mathbf{p}_{(\Gamma_G,\gamma)}\right) = []$$

and we'll end up with a very even partition whenever the integers m_i+1 are all even.

In fact, the only time we end up with an even-even partition coresponding to Γ is when there is a corresponding Γ' differing from Γ by the substitution $\alpha_{n-1} \to \alpha_n$. This will be but then, regarding α_n as contiguous with α_{n-2} (as it is in the Dynkin diagram of $\mathfrak{so}(2n)$), the same algorithm will produce the same even-even partition that is attached to the original Γ. It thus make sense to simply adopt the convention that the even-even partition attached to Γ (the one not involving α_n) is type I, while the even-even partition corresponding to Γ' (the one not involving α_{n-1}) is said to be of type II..

Example 2.16. $\mathfrak{so}(8)$

Diagram	$[\Gamma,\gamma]$	$\mathfrak{l}_\Gamma$	Γ_A	$\mathbf{m}_A$	Γ_G	$\left(\mathbf{p}_{(\Gamma_G,\gamma)}\right)$	$\mathbf{p}_{(\Gamma,\gamma)}$
	$[[],[]]$	$\mathbf{0}$	$[]$	$[]$	$[]$	$[]$	$\left[(1)^8\right]$
	$[[1],[]]$	A_1	$[1]$	$[2,2]$	$[]$	$[]$	$\left[(2^2),(1)^4\right]$
	$[[1,2],[]]$	A_2	$[1,2]$	$[3,3]$	$[]$	$[]$	$\left[(3)^2,(1)^2\right]$
	$[[1,3],[]]$	$2A_1$	$[1,3]$	$[2,2,2,2]$	$[]$	$[]$	$\left[(2)^4\right]^I$
	$[[1,4],[]]$	$2A_1$	$[1,4]$	$[2,2,2,2]$	$[]$	$[]$	$\left[(2)^4\right]^{II}$
	$[[3,4],[]]$	D_2	$[]$	$[]$	$[3,4]$	$[4]$	$\left[4,(1)^4\right]$
	$[[1,3,4],[]]$	A_1+D_2	$[1]$	$[2,2]$	$[3,4]$	$[4]$	$\left[4,(2)^2\right]$
	$[[1,2,3],[]]$	A_3	$[1,2,3]$	$[4,4]$	$[]$	$[]$	$\left[(4)^2\right]^I$
	$[[1,2,4],[]]$	A_3	$[1,2,4]$	$[4,4]$	$[]$	$[]$	$\left[(4)^2\right]^{II}$
	$[[2,3,4],[]]$	D_3	$[]$	$[]$	$[2,3,4]$	$[5,1]$	$\left[5,(1)^3\right]$
	$[[1,2,3,4],[2]]$	$D_4(a_1)$	$[]$	$[]$	$[1,2,3,4]$	$[5,3,1]$	$[5,3,1]$
	$[[1,2,3,4],[]]$	D_4	$[]$	$[]$	$[1,2,3,4]$	$[7,1]$	$[7,1]$

Remark 2.17. Actually, once one knows the partitions corresponding to the distinguished orbits of Lie algebras of same Cartan type as $\mathfrak{g}$ it is easy to go directly from the CBC-diagram for $[\Gamma,\gamma]$ to the corresponding partition. Below we give a table of the distinguished orbits for the classical Lie algebras up to rank 8. In fact, once one is able to recognize the partitions of distinguished orbits of the same Cartan type it is easy to extract the CBC-diagram corresponding to a given partition. The following two examples utilize the tables of CBC-diagrams and distinguished orbits given in the Appendix B.

Example 2.18. Find the partition corresponding to the CBC-diagram $\bullet - \bullet - \circ - \bullet - \circ - \bullet - * - \bullet \Longrightarrow *$. Well the A-head is

$$\bullet - \bullet - \circ - \bullet \quad \rightsquigarrow \quad A_2 + A_1 \quad \rightsquigarrow \quad [3,3,2,2]$$

and the G-tail is

$$\bullet - * - \bullet \Longrightarrow * \quad \rightsquigarrow \quad B_4(a_2) \quad \rightsquigarrow \quad [5,3,1]$$

So the corresponding partition should be

$$[5,3,1,3,3,2,2] \quad \rightsquigarrow \quad [5,3,3,3,2,2,1]$$

Example 2.19. Find the CBC diagram corresponding to the partition $[4,4,4,3,3,2,2,2] \in \mathcal{P}_{\mathfrak{sp}(24)}$. Well, we think of the parts as breaking up into pairs and singlets:

$$\begin{aligned}
[4,4,4,3,3,2,2,2] &\rightsquigarrow ([4,4]+[3,3]+[2,2])+([4]+[2]) \\
&\rightsquigarrow (A_3+A_2+A_1)+(C_3\,(a_1)) \\
&\rightsquigarrow \bullet - \bullet - \bullet - \circ - \bullet - \bullet - \circ - \bullet - * \Longleftarrow \bullet
\end{aligned}$$

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