A combinatorial parameterization of nilpotent orbits, twisted induction, and duality, II

O.S.U. Lie Groups Seminar
March 5, 2008

Last time we developed a parameterization of the nilpotent oribits of a complex semisimple Lie algebra $\mathfrak{g}$ in terms of certain pairs (Γ, γ); where Γ was a subset of simple roots (representating a conjugacy class $G \cdot \mathfrak{l}_\Gamma$ of Levi subalgebras) and γ was a distinguished subset of Γ (corresponding to distinguished L_Γ-orbit in $\mathfrak{l}_\Gamma$). The goal of today's talk is to use this formalism to pose a solution to a problem posed by David Vogan (how to define a natural duality map that does not utilize the dual Lie algebra or a choice of a nondegenerate invariant bilinear form) and to make contact with the work of George Lusztig on cells of Weyl group representations.

1. The Springer Correspondence

Suppose x is a nilpotent element of $\mathfrak{g}$. Then $u = \exp(x)$ is a unipotent element of G. Let $\mathfrak{B}_u$ be the (projective) variety of all Borel subgroups of G that contain u. Springer studied the cohomology groups $H^i(\mathfrak{B}_u, \mathbb{Q})$ and in the process showed how one could define an action of W, the Weyl group of $\mathfrak{g}$, on the top non-vanishing homology class of $H^{top}(\mathfrak{B}_x, \mathbb{Q})$. The centralizer G^u of u also acts on $H^{top}(\mathfrak{B}_x, \mathbb{Q})$ by conjugation and it turns out that the finite group

$$A(u) = G^u / (G^u)_0$$

acts on $V_u = H^{top}(\mathfrak{B}_x, \mathbb{Q})$ in such a way that its action commutes with that of W. In fact, Springer proves:

Theorem 1.1. *For each irreducible character ψ of $A(u)$ let $\psi_{u,\psi}$ be the direct sum of all $A(u)$-submodules of V_u upon which $A(u)$ acts via the character ψ. Then $V_{u,\psi}$, when non-empty, is a W-submodule of V_u. In fact,*

- (i) *$V_{u,\psi}$, when non-empty, is a direct sum of isomorphic irreducible W-modules*
- (ii) *Let $\sigma_{u,\psi}$ be the irreducible character of W corresponding to a $V_{u,\psi} \neq 0$. Then each irreducible character of W occurs as a $\sigma_{u,\psi}$ for some unipotent element $u \in G$ and some irreducible character ψ of $A(u)$.*
- (iii) *$\sigma_{u,\psi} = \sigma_{u',\psi'}$ if and only if u is conjugate to u' and $\psi = \psi'$*
- (iv) *Thus, the irreducible characters (and hence the irreducible representations) of W are parameterized by pairs $(G \cdot u, \psi)$ where $G \cdot u$ is a unipotent conjugacy class of G and ψ is an irreducible character of $A(u)$.*
- (v) *$V_{u,1}$, where 1 represents the trivial character of $A(u)$ is always non-zero.*

Thus, as a representation of $W \otimes A(u)$, the cohomology space $H^{top}(\mathfrak{B}_x, \mathbb{Q})$ decomposes as

$$V_u \approx m_1 \sigma_{u,1} \otimes 1 + m_2 \sigma_{u,\psi_{u,1}} \otimes \psi_{\mu,1} + \cdots + m_k \sigma_{u,\psi_{u,k}} \otimes \psi_{u,k}$$

The correspondence

(*) $$\mathcal{O} \ni x \to u \to V_u \to \phi_{u,1} \in \widehat{W}$$

is the famous Springer correspondence; it attaches to each nilpotent orbit a certain irreducible representation of the Weyl group. Of course, because of (iii), the correspondence (*) only establishes a 1:1 correspondence between nilpotent orbits and a certain subset of $\widehat{W}$. On the other hand, Theorem 1.1 can be viewed as providing a 1:1 correspondence between (equivalence classes of) irreducible representations of W and a certain family of local systems over nilpotent orbits.

2. Lusztig cells in $\widehat{W}$

In view of the Springer correspondence above, any way of organizing the representations in $\widehat{W}$ will lead to a certain way of organizing the nilpotent orbits. Below we shall introduce Lusztig's notion of cells in $\widehat{W}$ and display the celluar structure of $G\backslash\mathcal{N}$ that is carried over $\widehat{W}$ by the Springer correspondence. In the last section of this talk we will introduce the notion of twisted induction and show how it and the combinatorial parameterization of nilpotent orbits replicates the Lusztig-Springer organization of nilpotent orbits in a completely intrinsic manner.

2.1. **Truncated Induction.** Attached to each irreducible representation σ of a Weyl group W are a pair of polynomials with rational coefficients in a single indeterminant X. The first is the *fake degree polynomial* $P_\sigma(X)$.

$$P_\sigma(X) = \sum_{i\geq 0} m_i X^i$$

where m_i is the multiplicity with which the representation σ occurs in $\overline{S}_i(\mathfrak{h})$, where $\overline{S}_i(\mathfrak{h})$ is the set of homogeneous W-harmonic polynomials on $\mathfrak{h}^*$ of degree i.

The second polynomial is the *generic degree polynomial* $\widetilde{P}_\sigma(X)$. It's definition is much more elaborate. Let G be the adjoint Chevalley group over k (an algebraic closure of the prime field F_p) with root system Δ. Let $G(q)$ be the group of F_q-rational points of G, where F_q is a subfield of k with q elements. Fix a homomorphism

$$h : \mathbb{C}[X]^* \to \mathbb{C}$$

where $\mathbb{C}[X]^*$ is the integral closure of $\mathbb{C}[X]$) such that $h(X) = q$. It is known that h gives rise to a $1:1$ correspondence $\sigma \to \pi_\sigma$ between the set of isomorphism classes of irreducible representations of W and the set of isomorphism classes of irreducible representations of $G(q)$ occuring in $Ind_{B(q)}^{G(q)}(\mathbf{1})$ (where B is a Borel subgroup of G defined over F_q and $B(q)$ is its group of F_q-rational points). The dimension The dimension of π_σ is independent of the choice of h and, in fact, corresponds to the evaluation at q of a polynomial $\widetilde{P}_\sigma(X)$. We write

$$\begin{aligned} P_\sigma(X) &= \gamma_\sigma X^{a_\sigma} + \cdots + \delta_\sigma X^{b_\sigma} \qquad , \qquad a_\sigma < b_\sigma \\ \widetilde{P}_\sigma(X) &= \widetilde{\gamma}_\sigma X^{\widetilde{a}_\sigma} + \cdots + \widetilde{\delta}_\sigma X^{\widetilde{b}_\sigma} \qquad , \qquad \widetilde{a}_\sigma < \widetilde{b}_\sigma \end{aligned}$$

(singling out the terms of lowest and highest degree).

Remark 2.1. Weyl group representations σ for which $a_\sigma = \widetilde{a}_\sigma$ are called **special** by Lusztig.

Next, Lusztig defines *cell representations* (actually, Lusztig refers to these representations simply as *cells*). Let W_Γ be a standard parabolic subgroup of W (i.e., a subgroup of W generated by the reflections of a subset Γ of the simple roots Π in Δ), and let σ' be an irreducible W_Γ module. Set

$$J_{W_\Gamma}^W(\sigma') = \sum \left\langle \sigma', Ind_{W_\Gamma}^W(\sigma')\right\rangle \sigma$$

where the sum runs over all $\sigma \in \widehat{W}$ such that $\widetilde{a}_\sigma = \widetilde{a}_{\sigma'}$. Extend the operation $J_{W_\Gamma}^W$ by linearity so that it may be applied to reducible representations.

Definition 2.2. *If $W = \{e\}$, then there is only one **cell representation**, the unit representation of W. Assume now that $W \neq \{e\}$ and that for any $\Gamma \subset \Pi$, the cells of W_Γ have been defined. The **cell representations** of W are the (not necessarily irreducible) representations of W of the form $J_{W_\Gamma}^W(c)$ and those of the form $J_{W_\Gamma}^W(c) \otimes sgn(W)$, where Γ runs over the subsets of Π and c runs over the cell representations of W_Γ.*

Fact 2.3.
- *Every irreducible representation of W appears as a component of some cell representation*
- *Every cell representation contains a unique special representation with multiplicity 1.*

- *Two cell representations have a common irrreducible component if and only if they have the same special component.*

We can thus, partition $\widehat{W}$ up into equivalence classes that we haven't yet figured out a good name for (left-cells might be the appropriate thing). At any rate, we'll call the collection of irreducible representations of W that appear in a cell representation an L-cell. Each L-cell then contains exactly one special representation.

Definition 2.4. *Let $\sigma, \sigma' \in \widehat{W}$. We shall say $\sigma \sim \sigma'$ if both σ and σ' occur in the same cell representation. $\sim$ is then an equivalence relation on the set $\widehat{W}$. We shall refer to these equivalence classes as L-cells in $\widehat{W}$.*

Remark 2.5. Note each L-cell contains exactly one special representation of $\widehat{W}$

The partitioning of $\widehat{W}$ into Lusztig cells induces via the Springer correspondence a certain partitioning of $G\backslash\mathcal{N}$. Let $s : G\backslash\mathcal{N} \longrightarrow \widehat{W}$ be the Springer correspondence. We shall say that

$$\mathcal{O} \sim \mathcal{O}' \text{ if } s(\mathcal{O}) \sim s(\mathcal{O}') \in \widehat{W}$$

and call the corresponding equivalence classes *LS-cells* in $G\backslash\mathcal{N}$.

In Appendix A we provide tables showing the L-cells for the exceptional Lie algebras. In these tables, the irreducible representations of the Weyl groups are specified in two different ways: the notation $\phi_{d,j}$ indicates the irreducible representation of W of dimension d that first occurs in $S(\mathfrak{h})$ in degree j. (This data does not always characterize the representation uniquely - when it does not we embellish $\phi_{d,j}$ with primes ($\phi'_{d,j}, \phi''_{d,j}$, etc.) to distinguish the irreducibles with the same dimension and degree attributes.. We also provide in those table the Springer parameters of the irreducible representations of W: Recall that the Springer correspondence attaches to a nilpotent orbit $\mathcal{O}$ the representation $\sigma \in \widehat{W}$ whose "orbit" parameter is $\mathcal{O}$ and whose $\widehat{A}(u)$ parameter is 1. Thus, by simply ignoring the rows of a table containing non-trivial $\widehat{A}(u)$ entries we can read off the LS-cells in $G\backslash\mathcal{N}$.

2.2. **Barbasch-Vogan duality.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{g}^\vee$ be its dual Lie algebra. Suppose $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$. In [BV], Barbasch and Vogan define a "duality map" $\eta_{\mathfrak{g}} : G\backslash\mathcal{N}_{\mathfrak{g}} \longrightarrow G^\vee\backslash\mathcal{N}_{\mathfrak{g}^\vee}$ with the following properties:

- $\eta_{\mathfrak{g}}$ is an order-reversing: if $\mathcal{O} \subset \mathcal{O}'$, then $\eta_{\mathfrak{g}}(\mathcal{O}') \subset \eta_{\mathfrak{g}}(\mathcal{O})$
- Reversing the roles of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ one has a map $\eta_{\mathfrak{g}^\vee} : G^\vee\backslash\mathcal{N}_{\mathfrak{g}^\vee} \longrightarrow G\backslash\mathcal{N}_{\mathfrak{g}}$ and moreover

$$\eta_{\mathfrak{g}} \circ \eta_{\mathfrak{g}^\vee} \circ \eta_{\mathfrak{g}} = \eta_{\mathfrak{g}}$$

- Suppose $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$, $\mathfrak{l}^\vee$ is a Levi subalgebra of $\mathfrak{g}^\vee$ dual to $\mathfrak{l}^\vee$ and $\mathcal{O}_{\mathfrak{l}^\vee}$ is a nilpotent orbit in $\mathfrak{l}^\vee$. Then

$$\eta_{g^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}(\eta_{\mathfrak{l}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})) \tag{2}$$

Definition 2.6. *A* special nilpotent orbit *is an orbit $\mathcal{O} \in G\backslash\mathcal{N}$ that lies in the image of $\eta_{\mathfrak{g}^\vee} : G^\vee\backslash\mathcal{N}_{\mathfrak{g}^\vee} \longrightarrow G\backslash\mathcal{N}$. We shall denote by $\mathcal{S}_{\mathfrak{g}}$ the set of special nilpotent orbits for $\mathfrak{g}$.*

Now suppose $\mathcal{O}_{l^\vee}$ is a distinguished orbit in Levi subalgebra $\mathfrak{l}^\vee$ of $\mathfrak{g}^\vee$. That is to say, $\mathcal{O}_{\mathfrak{l}^\vee}$ is the Richardson orbit corresponding to (the Levi factor of) some distinguished parabolic subalgebra of $\mathfrak{l}^\vee$. Letting $(\mathfrak{l}^\vee, \mathcal{O}_{\mathfrak{l}^\vee})$ run though the $G^\vee$-conjugacy classes of all such pairs, $inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}(\mathcal{O}_{\mathfrak{l}^{\vee e}})$ hits every nilpotent orbit of $\mathfrak{g}^\vee$. Thus,

$$\mathcal{S}_{\mathfrak{g}} = \left\{\eta_{g^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})\right) \mid (\mathfrak{l}^\vee, \mathcal{O}_{\mathfrak{v}^\vee}) \text{ a Bala-Carter pair for } \mathfrak{g}^\vee\right\} \tag{3}$$

2.3. **Intrinsic duality.** The construction of the special orbits of $\mathfrak{g}$ via the duality map $\eta_{\mathfrak{g}^\vee}$ hints at an inductive approach to duality that does not require the auxiliary apparatus of dual Lie algebras. Indeed, inserting the right-hand-side of (2) into (3) yields

$$\mathcal{S}_{\mathfrak{g}} = \left\{ ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\eta_{\mathfrak{l}^\vee} \left(\mathcal{O}_{\mathfrak{l}^\vee} \right) \right) \mid \left(\mathfrak{l}^\vee, \mathcal{O}_{\mathfrak{v}^\vee} \right) \text{ a Bala-Carter pair for } \mathfrak{g}^\vee \right\}$$

That is to say,

Observation 2.7. Every special orbit is induced from the dual of a distinguished orbit of a Levi subalgebra of $\mathfrak{g}^\vee$.

Next we note that if $(\mathfrak{l}^\vee, \mathcal{O}_{\mathfrak{v}^\vee})$ is a Bala-Carter pair for $\mathfrak{g}^\vee$, then $\eta_{\mathfrak{l}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})$ is a special orbit in $\mathfrak{l}$ (obviously, it lies in the image of $\eta_{\mathfrak{l}^\vee}$). Thus, every special orbit in $\mathfrak{g}$ is induced from a special orbit in a Levi subalgebra which is the dual of distinguished orbit $\mathcal{O}_{\mathfrak{l}^\vee}$. The goal of this section, is to replace the preceding statement with the following:

Every special orbit in $\mathfrak{g}$ is induced from the dual of a disinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.

This statement is in fact easily proved if one takes the dual $d(\mathcal{O}_{\mathfrak{l}})$ of a distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ to be the special nilpotent orbit in $\mathfrak{g}$ (in the sense of Luzstig) whose Springer representation in $\widehat{W}$ coincides with that of $\eta_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}}) \in \mathcal{S}_{\mathfrak{l}^\vee}$. But this definition of an intrinsic duality map $d : \mathcal{S}_{\mathfrak{l}} \to \mathcal{S}_{\mathfrak{l}}$ would just sweep the dual Lie algebra aspects of the construction under the rug.

Before preceding further we should note every distinguished orbit is also *special* in the sense of Spaltenstein. This can be verified by a case-by-case inspection of the exceptional Lie algebras and by comparing the partition criteria for special orbits (pg. 101 of [CM]) with the paritition criteria (pg 126 of [CM]) for distiguished orbits.

Shortly, we shall give a uniform definition of a duality map $d : G\backslash\mathcal{N} \to \mathcal{S}_{\mathfrak{g}}$ that coincides with that of Spaltenstein. The formulation of this definition, however, will be inductive. We shall see later that the inductive nature of this definition makes it particularly suitable for making contact with Luzstig's construction of Weyl group cells.

Definition 2.8 (Part 1)**.** *Suppose, as an inductive hypothesis, we have defined the dual $d(\mathcal{O}_{\mathfrak{l}}) \in \mathcal{S}_{\mathfrak{l}}$ of any distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in any proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and let $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ be the Bala-Carter parameters of a nilpotent orbit $\mathcal{O}$ that is **not** distinguished in $\mathfrak{g}$ (in other words, we assume $\mathfrak{l}$ is a proper subalgebra of $\mathfrak{g}$). Then we defined the dual of $\mathcal{O}$ to be*

$$d(\mathcal{O}) = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(d(\mathcal{O}_{\mathfrak{l}}) \right)$$

We note that $d(\mathcal{O})$ is well-defined modulo our inductive hypothesis. Furthermore, $d(\mathcal{O})$ will always be a special orbit in the sense of Spaltenstein because induction preserves "special-ness" (ref). To seal the deal we need a definition of duality for distinguished orbits.

Remark 2.9.

- Suppose $\mathcal{O}$ is distinguished in $\mathfrak{g}$. Then $\mathcal{O}$ may be viewed as the Richardson orbit attached to a distinguished subset Γ of Π :

$$\mathcal{O} = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} (\mathbf{0})$$

  But the trivial orbit of $\mathfrak{l}_\Gamma$ is always dual to the principal orbit $\mathcal{O}_{\mathfrak{l},prin}$ of $\mathfrak{l}_\Gamma$. And the principal orbit of $\mathfrak{l}_\Gamma$ is always distinguished in $\mathfrak{l}_\Gamma$. Thus, if $\mathcal{O}$ is distinguished

$$\mathcal{O} = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} (\mathbf{0}) = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} \left(d \left(\mathcal{O}_{\mathfrak{l}_\Gamma,prin} \right) \right)$$

  and so the set

$$\left\{ \mathcal{O}_{(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})} \mid \mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(d(\mathcal{O}_{\mathfrak{l}}) \right) \right\}$$

  is always non-empty.
- It is an empirical fact that the set $\left\{ \mathcal{O}_{(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})} \mid \mathcal{O} = \widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}} (\mathcal{O}_{\mathfrak{l}}) \right\}$ has a unique maximal element which is also a special orbit in the sense of Spaltenstein (and in fact, corresponds to a special representation of the Weyl group via the Springer correspndence).

Definition 2.10 (Part 2)**.** *Let $\mathcal{O}$ is a distinguished orbit in $\mathfrak{g}$. The dual $d(\mathcal{O})$ of $\mathcal{O}$ is*

$$d(\mathcal{O}) \equiv \max\left\{\mathcal{O}_{(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})} \mid \mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)\right\}$$

That is to say, $d(\mathcal{O})$ is the largest orbit (w.r.t. the closure partial ordering of $G\backslash\mathcal{N}$) from which $\mathcal{O}$ can be obtained by twisted induction.

Parts 1 and 2, now provide an intrinsic definition of the duality map $d: G\backslash\mathcal{N} \to \mathcal{S}_{\mathfrak{g}}$. The construct $(\mathfrak{l},\mathcal{O}_{\mathfrak{l}}) \to ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)$ is also worthy of a title of its own.

Definition 2.11. *The nilpotent orbit in $\mathcal{S}_{\mathfrak{g}}$ obtained from a distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ by* twisted induction *is the orbit*

$$\widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) \equiv ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right) \quad .$$

Theorem 2.12.
- *Every special orbit in $\mathfrak{g}$ is obtained by twisted induction from a distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.*
- *Let $\sim$ be the equivalence relation on $G\backslash\mathcal{N}$ defined by*

$$\mathcal{O}_{(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})} \sim \mathcal{O}'_{\left(\mathfrak{l}',\mathcal{O}'_{\mathfrak{l}}\right)} \quad \Longleftrightarrow \quad \widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) = \widetilde{ind}_{\mathfrak{l}'}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}'}\right)$$

Then the corresponding partitioning of $G\backslash\mathcal{N}$ coincides precisely with the partitioning of $G\backslash\mathcal{N}$ by LS-cells.

References

[BV] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. of Math. **121** (1985), 41-110.
[C] R. Carter, *Finite Groups of Lie Type*, John Wiley and Sons, New York, 1985.
[CW] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, New York (1993).
[L1] G. Lusztig, *A class of irreducible representations of a Weyl group*, Indag. Math. **41** (1979), 323-335.
[L2] G. Lusztig, *A class of irreducible representations of a Weyl group, II*, Indag. Math. **44** (1982), 219-226.
[Sp] N. Spaltenstein, *A property of special representations of Weyl groups*, J. Reine Angew. Math. **343** (1983), 212-220.
[Spr] T.A. Springer, *A construction of representations of Weyl groups*, Inv. Math. **44** (1978). 279-293.
[Vo] D. Vogan (Atlas Wiki entry, August 2007).

Appendix A. W-Reps, Lusztig Cells, and Orbits

G_2

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	G_2	1	1
$\phi_{2,1}$	$G_2(a_1)$	S_3	1
$\phi'_{1,3}$	$G_2\,(a_1)$	S_3	ψ_{21}
$\phi_{2,2}$	$\tilde{A}_1$	1	1
$\phi''_{1,3}$	A_1	1	1
$\phi 1, 6$	$\mathbf{0}$	1	1

F_4

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	F_4	1	1
$\phi_{4,1}$	$F_4(a_1)$	S_2	1
$\phi'_{2,4}$	$F_4\,(a_1)$	S_2	ψ_{11}
$\phi''_{2,4}$	$F_4\,(a_2)$	S_2	ψ_{11}
$\phi_{9,2}$	$F_4(a_2)$	S_2	1
$\phi'_{8,3}$	C_3	1	1
$\phi''_{8,3}$	B_3	1	1
$\phi_{12,4}$	$F_4(a_3)$	S_4	1
$\phi''_{6,6}$	$F_4\,(a_3)$	S_4	ψ_{22}
$\phi'_{9,6}$	$F_4\,(a_3)$	S_4	ψ_{31}
$\phi'_{1,12}$	$F_4\,(a_3)$	S_4	ψ_{211}
$\phi_{16,5}$	$C_3(a_1)$	S_2	1
$\phi'_{4,7}$	$C_3\,(a_1)$	S_2	ψ_{11}
$\phi'_{6,6}$	$\widetilde{A}_2 + A_1$	1	1
$\phi''_{9,6}$	B_2	S_2	1
$\phi_{4,8}$	B_2	S_2	ψ_{11}
$\phi''_{4,7}$	$A_2 + \widetilde{A}_1$	1	1
$\phi''_{1,12}$	A_2	S_2	ψ_{11}
$\phi'_{8,9}$	$\widetilde{A}_2$	1	1
$\phi''_{8,9}$	A_2	S_2	1
$\phi_{9,10}$	$\widetilde{A}_1 + A_1$	1	1
$\phi_{4,13}$	$\widetilde{A}_1$	S_2	1
$\phi'_{2,16}$	$\widetilde{A}_1$	S_2	ψ_{11}
$\phi''_{2,16}$	A_1	1	1
$\phi_{1,24}$	$\mathbf{0}$	1	1

E_6

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	E_6	1	1
$\phi_{6,1}$	$E_6\,(a_1)$	1	1
$\phi_{20,2}$	D_5	1	1
$\phi_{30,3}$	$E_6\,(a_3)$	S_2	1
$\phi_{15,5}$	$E_6\,(a_3)$	S_2	ψ_{11}
$\phi_{15,4}$	A_5	1	1
$\phi_{64,4}$	$D_5\,(a_1)$	1	1
$\phi_{60,5}$	$A_4 + A_1$	1	1
$\phi_{24,6}$	D_4	1	1
$\phi_{81,6}$	A_4	1	1
$\phi_{80,7}$	$D_4\,(a_1)$	S_3	1
$\phi_{90,8}$	$D_4\,(a_1)$	S_3	ψ_{21}
$\phi_{20,10}$	$D_4\,(a_1)$	S_3	ψ_{111}
$\phi_{60,8}$	$A_3 + A_1$	1	1
$\phi_{10,9}$	$2A_2 + A_1$	1	1
$\phi_{81,10}$	A_3	1	1
$\phi_{60,11}$	$A_2 + 2A_1$	1	1
$\phi_{24,12}$	$2A_2$	1	1
$\phi_{64,13}$	$A_2 + A_1$	1	1
$\phi_{30,15}$	A_2	S_2	1
$\phi_{15,17}$	A_2	S_2	ψ_{11}
$\phi_{15,16}$	$3A_1$	1	1
$\phi_{20,20}$	$2A_1$	1	1
$\phi_{6,25}$	A_1	1	1
$\phi_{1,36}$	$\mathbf{0}$	1	1

E_7

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	E_7	1	1
$\phi_{7,1}$	$E_7(a_1)$	1	1
$\phi_{27,2}$	$E_7(a_2)$	1	1
$\phi_{56,3}$	$E_7(a_3)$	S_2	1
$\phi_{21,6}$	$E_7\,(a_3)$	S_2	ψ_{11}
$\phi_{35,4}$	D_6	1	1
$\phi_{21,3}$	E_6	1	1
$\phi_{120,4}$	$E_6(a_1)$	S_2	1
$\phi_{105,5}$	$E_6\,(a_1)$	S_2	ψ_{11}
$\phi_{15,7}$	$E_7\,(a_4)$	S_2	ψ_{11}
$\phi_{189,5}$	$E_7(a_4)$	S_2	1
$\phi_{210,6}$	$D_6(a_1)$	1	1
$\phi_{168,6}$	$D_5 + A_1$	1	1
$\phi_{105,6}$	A_6	1	1
$\phi_{315,7}$	$E_7(a_5)$	S_3	1
$\phi_{280,9}$	$E_7\,(a_5)$	S_3	ψ_{21}
$\phi_{35,13}$	$E_7\,(a_5)$	S_3	ψ_{111}
$\phi_{280,8}$	$D_6(a_2)$	1	1
$\phi_{70,9}$	$A_5 + A_1$	1	1
$\phi_{189,7}$	D_5	1	1
$\phi_{405,8}$	$E_6(a_3)$	S_2	1
$\phi_{189,10}$	$E_6\,(a_3)$	S_2	ψ_{11}
$\phi_{216,9}$	$(A_5)'$	1	1
$\phi_{378,9}$	$D_5(a_1) + A_1$	1	1
$\phi_{210,10}$	$A_4 + A_2$	1	1
$\phi_{420,10}$	$D_5(a_1)$	S_2	1
$\phi_{336,11}$	$D_5\,(a_1)$	S_2	ψ_{11}
$\phi_{84,12}$	$D_4 + A_1$	1	1
$\phi_{512,11}$	$A_4 + A_1$	S_2	1
$\phi_{512,12}$	$A_4 + A_1$	S_2	ψ_{11}

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{105,12}$	$(A_5)''$	1	1
$\phi_{210,13}$	$A_3 + A_2 + A_1$	1	1
$\phi_{420,13}$	A_4	S_2	1
$\phi_{336,14}$	A_4	S_2	ψ_{11}
$\phi_{84,15}$	$A_3 + A_2$	S_2	ψ_{11}
$\phi_{378,14}$	$A_3 + A_2$	S_2	1
$\phi_{405,15}$	$D_4(a_1) + A_1$	S_2	1
$\phi_{189,17}$	$D_4\,(a_1) + A_1$	S_2	ψ_{11}
$\phi_{216,16}$	$A_3 + 2A_1$	1	1
$\phi_{105,15}$	D_4	1	1
$\phi_{315,16}$	$D_4(a_1)$	S_3	1
$\phi_{280,18}$	$D_4\,(a_1)$	S_3	ψ_{21}
$\phi_{35,22}$	$D_4\,(a_1)$	S_3	ψ_{111}
$\phi_{280,17}$	$(A_3 + A_1)'$	1	1
$\phi_{70,18}$	$2A_2 + A_1$	1	1
$\phi_{189,20}$	$(A_3 + A_1)''$	1	1
$\phi_{105,21}$	$A_2 + 3A_1$	1	1
$\phi_{168,21}$	$2A_2$	1	1
$\phi_{210,21}$	A_3	1	1
$\phi_{189,22}$	$A_2 + 2A_1$	1	1
$\phi_{120,25}$	$A_2 + A_1$	S_2	1
$\phi_{105,26}$	$A_2 + A_1$	S_2	ψ_{11}
$\phi_{15,28}$	$4A_1$	1	1
$\phi_{56,30}$	A_2	S_2	1
$\phi_{21,33}$	A_2	S_2	ψ_{11}
$\phi_{35,31}$	$(3A_1)'$	1	
$\phi_{21,36}$	$(3A_1)''$	1	
$\phi_{27,37}$	$2A_1$	1	
$\phi_{7,46}$	A_1	1	
$\phi_{1,63}$	$\mathbf{0}$	1	

E_8

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	E_8	1	1
$\phi_{8,1}$	$E_8\,(a_1)$	1	1
$\phi_{35,2}$	$E_8\,(a_2)$	1	1
$\phi_{112,3}$	$E_8\,(a_3)$	S_2	1
$\phi_{210,4}$	$E_8\,(a_4)$	S_2	1
$\phi_{160,7}$	$E_8\,(a_4)$	S_2	ψ_{11}
$\phi_{50,8}$	$E_8\,(b_4)$	S_2	ψ_{11}
$\phi_{84,4}$	E_7	1	1
$\phi_{28,8}$	$E_8\,(a_3)$	S_2	ψ_{11}
$\phi_{560,5}$	$E_8\,(b_4)$	S_2	1
$\phi_{700,6}$	$E_8\,(a_5)$	S_2	1
$\phi_{300,8}$	$E_8\,(a_5)$	S_2	ψ_{11}
$\phi_{400,7}$	D_7	1	1
$\phi_{567,6}$	$E_7\,(a_1)$	1	1
$\phi_{1400,7}$	$E_8\,(b_5)$	S_3	1
$\phi_{1400,8}$	$E_8\,(a_6)$	S_3	1
$\phi_{1575,10}$	$E_8\,(a_6)$	S_3	ψ_{21}
$\phi_{1050,10}$	$D_7\,(a_1)$	S_2	ψ_{11}
$\phi_{350,14}$	$E_8\,(a_6)$	S_3	ψ_{111}
$\phi_{175,12}$	$E_8\,(b_6)$	S_3	ψ_{21}
$\phi_{1344,8}$	$E_7\,(a_2)$	1	1
$\phi_{448,9}$	$E_6 + A_1$	1	1
$\phi_{1008,9}$	$E_8\,(b_5)$	S_3	ψ_{21}
$\phi_{56,19}$	$E_8\,(b_5)$	S_3	ψ_{111}
$\phi_{3240,9}$	$D_7\,(a_1)$	S_2	1
$\phi_{2240,10}$	$E_8\,(b_6)$	S_3	1
$\phi_{4096,11}$	$E_6\,(a_1) + A_1$	S_2	1
$\phi_{4096,12}$	$E_6\,(a_1) + A_1$	S_2	ψ_{11}
$\phi_{4200,12}$	$D_7\,(a_2)$	S_2	1
$\phi_{3360,13}$	$D_7\,(a_2)$	S_2	ψ_{11}
$\phi_{840,14}$	$D_5 + A_2$	S_2	ψ_{11}
$\phi_{1400,11}$	A_7	1	1
$\phi_{840,13}$	$E_8\,(b_6)$	S_3	ψ_{111}
$\phi_{2268,10}$	$E_7\,(a_3)$	S_2	1
$\phi_{1296,13}$	$E_7\,(a_3)$	S_2	ψ_{11}
$\phi_{972,12}$	D_6	1	1
$\phi_{525,12}$	E_6	1	1
$\phi_{4536,13}$	$D_5 + A_2$	S_2	1
$\phi_{2800,13}$	$E_6\,(a_1)$	S_2	
$\phi_{2100,16}$	$E_6\,(a_1)$	S_2	ψ_{11}
$\phi_{700,16}$	$E_7\,(a_4)$	S_2	ψ_{11}
$\phi_{2835,14}$	$A_6 + A_1$	1	1
$\phi_{6075,14}$	$E_7\,(a_4)$	S_2	1
$\phi_{5600,15}$	$D_6\,(a_1)$	S_2	1

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{4480,16}$	$E_8\,(a_7)$	S_5	1
$\phi_{4536,18}$	$E_8\,(a_7)$	S_4	ψ_{32}
$\phi_{5670,18}$	$E_8\,(a_7)$	S_5	ψ_{41}
$\phi_{1680,22}$	$E_8\,(a_7)$	S_5	ψ_{311}
$\phi_{1400,20}$	$E_8\,(a_7)$	S_5	ψ_{221}
$\phi_{70,32}$	$E_8\,(a_7)$	S_5	ψ_{2111}
$\phi_{7168,17}$	$E_7\,(a_5)$	S_3	1
$\phi_{5600,19}$	$E_7\,(a_5)$	S_3	ψ_{21}
$\phi_{448,25}$	$E_7\,(a_5)$	S_3	ψ_{111}
$\phi_{3150,18}$	$E_6\,(a_3) + A_1$	S_2	1
$\phi_{1134,20}$	$E_6\,(a_3) + A_1$	S_2	ψ_{11}
$\phi_{4200,18}$	$D_6\,(a_2)$	S_2	1
$\phi_{2688,20}$	$D_6\,(a_2)$	S_2	ψ_{11}
$\phi_{1344,19}$	$D_5\,(a_1) + A_2$	1	1
$\phi_{2016,19}$	$A_5 + A_1$	1	1
$\phi_{420,20}$	$A_4 + A_3$	1	1
$\phi_{168,24}$	$D_4 + A_2$	S_2	ψ_{11}
$\phi_{3200,16}$	$D_5 + A_1$	1	1
$\phi_{2400,17}$	$D_6\,(a_1)$	S_2	ψ_{11}
$\phi_{4200,15}$	A_6	1	1
$\phi_{2100,20}$	D_5	1	1
$\phi_{5600,21}$	$E_6\,(a_3)$	S_2	1
$\phi_{2400,23}$	$E_6\,(a_3)$	S_2	ψ_{11}
$\phi_{3200,22}$	A_5	1	1
$\phi_{4200,21}$	$D_4 + A_2$	S_2	1
$\phi_{2835,22}$	$A_4 + A_2 + A_1$	1	1
$\phi_{6075,22}$	$D_5\,(a_1) + A_1$	1	1
$\phi_{4536,23}$	$A_4 + A_2$	1	1
$\phi_{4200,24}$	$A_4 + 2A_1$	S_2	1
$\phi_{2800,25}$	$D_5\,(a_1)$	S_2	1
$\phi_{2100,28}$	$D_5\,(a_1)$	S_2	ψ_{11}
$\phi_{700,28}$	$D_4 + A_1$	1	1
$\phi_{840,26}$	$2A_3$	S_2	1
$\phi_{3360,25}$	$A_4 + 2A_1$	S_2	ψ_{11}
$\phi_{4096,26}$	$A_4 + A_1$	S_2	1
$\phi_{4096,27}$	$A_4 + A_1$	S_2	ψ_{11}
$\phi_{2240,28}$	$D_4\,(a_1) + A_2$	S_2	1
$\phi_{840,31}$	$D_4\,(a_1) + A_2$	S_2	ψ_{11}
$\phi_{1400,29}$	$A_3 + A_2 + A_1$	1	1
$\phi_{2268,30}$	A_4	S_2	1
$\phi_{1296,33}$	A_4	S_2	ψ_{11}
$\phi_{972,32}$	$A_3 + A_2$	S_2	ψ_{11}
$\phi_{3240,31}$	$A_3 + A_2$	S_2	1

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1400,32}$	$D_4\,(a_1) + A_1$	S_3	1
$\phi_{1575,34}$	$D_4\,(a_1) + A_1$	S_3	ψ_{21}
$\phi_{350,38}$	$D_4\,(a_1) + A_1$	S_3	ψ_{111}
$\phi_{1050,34}$	$A_3 + 2A_1$	1	1
$\phi_{175,36}$	$2A_2 + 2A_1$	1	1
$\phi_{525,36}$	D_4	1	1
$\phi_{1400,37}$	$D_4\,(a_1)$	S_3	1
$\phi_{1008,39}$	$D_4\,(a_1)$	S_3	ψ_{21}
$\phi_{1344,38}$	$A_3 + A_1$	1	1
$\phi_{448,39}$	$2A_2 + A_1$	1	1
$\phi_{56,49}$	$D_4\,(a_1)$	S_3	ψ_{111}
$\phi_{700,42}$	$2A_2$	S_2	1
$\phi_{300,44}$	$2A_2$	S_2	ψ_{11}
$\phi_{400,43}$	$A_2 + 3A_1$	1	1
$\phi_{567,46}$	A_3	1	1
$\phi_{560,47}$	$A_2 + 2A_1$	1	1
$\phi_{210,52}$	$A_2 + A_1$	1	1
$\phi_{50,56}$	$4A_1$	1	1
$\phi_{112,63}$	A_2	S_2	1
$\phi_{28,68}$	A_2	S_2	ψ_{11}
$\phi_{84,64}$	$3A_1$	1	1
$\phi_{160,55}$	$A_2 + A_1$	S_2	ψ_{11}
$\phi_{35,74}$	$2A_1$	1	1
$\phi_{8,91}$	A_1	1	1
$\phi_{1,120}$	$\mathbf{0}$	1	1

Appendix B. Twisted Induction and LS-Cells

G_2

$\mathcal{O}$	$\mathfrak{l}$	$\mathcal{O}_{\mathfrak{l}}$	$\mathcal{O}_{\mathfrak{l}}^{\vee}$	$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}^{\vee}\right)$
G_2	G_2	G_2	$\mathbf{0}$	$\mathbf{0}$
$G_2(a_1)$	G_2	$G_2\left(a_1\right)$	$G_2\left(a_1\right)$	$G_2\left(a_1\right)$
$\tilde{A}_1$	$\widetilde{A_1}$	$[2]$	$\left[1^2\right]$	$G_2\left(a_1\right)$
A_1	A_1	$[2]$	$\left[1^2\right]$	$G_2\left(a_1\right)$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	G_2

F_4

$\mathcal{O}$	$\mathfrak{l}$	$\mathcal{O}_{\mathfrak{l}}$	$\mathcal{O}_{\mathfrak{l}}^{\vee}$	$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}^{\vee}\right)$
F_4	F_4	F_4	$\mathbf{0}$	$\mathbf{0}$
$F_4(a_1)$	F_4	$F_4\left(a_1\right)$	$\widetilde{A}_1$	$\widetilde{A}_1$
$F_4(a_2)$	F_4	$F_4\left(a_2\right)$	$\widetilde{A}_1+A_1$	$\widetilde{A_1+A_1}$
C_3	C_3	$[6]$	$\mathbf{0}$	A_2
B_3	B_3	$[7]$	$\mathbf{0}$	$\widetilde{A}_2$
$F_4(a_3)$	F_4	$F_4\left(a_3\right)$	$F_4\left(a_3\right)$	$F_4\left(a_3\right)$
$C_3(a_1)$	C_3	$[4,2]$	$\left[2^2,1^2\right]=ind_{C_2\approx B_2}^{C_3}\left(\mathbf{0}\right)$	$F_4\left(a_3\right)=ind_{C_3}^{F_4}\left(ind_{B_2}^{C_3}\left(\mathbf{0}\right)\right)$
$\widetilde{A}_2+A_1$	$\widetilde{A}_2+A_1$	$[3]+[2]$	$\mathbf{0}$	$F_4\left(a_3\right)$
B_2	B_2	$[5]$	$\mathbf{0}$	$F_4\left(a_3\right)$
$A_2+\widetilde{A}_1$	$A_2+\widetilde{A}_1$	$[3]+[2]$	$\mathbf{0}$	$F_4\left(a_3\right)$
$\widetilde{A}_2$	$\widetilde{A}_2$	$[3]$	$\mathbf{0}$	B_3
A_2	A_2	$[3]$	$\mathbf{0}$	C_3
$\widetilde{A}_1+A_1$	$\widetilde{A}_1+A_1$	$[2]+[2]$	$\mathbf{0}$	$F_4\left(a_2\right)$
$\widetilde{A}_1$	$\widetilde{A}_1$	$[2]$	$\mathbf{0}$	$F_4\left(a_1\right)$
A_1	A_1	$[2]$	$\mathbf{0}$	$F_4\left(a_1\right)$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	F_4	F_4

E_6

$\mathcal{O}$	$\mathfrak{l}$	$\mathcal{O}_{\mathfrak{l}}$	$\mathcal{O}_{l}^{\vee}$	$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}^{\vee}\right)$
E_6	E_6	E_6	$\mathbf{0}$	$\mathbf{0}$
$E_6\left(a_1\right)$	E_6	$E_6\left(a_1\right)$	A_1	A_1
D_5	D_5	$[9,1]$	$\left[1^{10}\right]$	$2A_1$
$E_6\left(a_3\right)$	E_6	$E_6\left(a_3\right)$	A_2	A_2
A_5	A_5	$[6]$	$\left[1^{6}\right]$	A_2
$D_5\left(a_1\right)$	D_5	$[7,5]$	$\left[2^2,1^{6}\right]$	A_2+A_1
A_4+A_1	A_4+A_1	$[5]+[2]$	$\left[1^{5}\right]+\left[1^{2}\right]$	A_2+2A_1
D_4	D_4	$[7,1]$	$\left[1^{8}\right]$	$2A_2$
A_4	A_4	$[5]$	$\left[1^{5}\right]$	A_3
$D_4\left(a_1\right)$	D_4	$[5,3]$	$\left[2^2,1^{4}\right]$	$D_4\left(a_1\right)$
A_3+A_1	A_3+A_1	$[4]+[2]$	$\left[1^{4}\right]+\left[1^{2}\right]$	$D_4\left(a_1\right)$
$2A_2+A_1$	$2A_2+A_1$	$[3]+[3]+[2]$	$\left[1^{3}\right]+\left[1^{3}\right]+\left[1^{2}\right]$	$D_4\left(a_1\right)$
A_3	A_3	$[4]$	$\left[1^{4}\right]$	A_4
A_2+2A_1	A_2+2A_1	$[3]+[2]+[2]$	$\left[1^{3}\right]+\left[1^{2}\right]+\left[1^{2}\right]$	A_4+A_1
$2A_2$	$2A_2$	$[3]+[3]$	$\left[1^{3}\right]+\left[1^{3}\right]$	D_4
A_2+A_1	A_2+A_1	$[3]+[2]$	$\left[1^{3}\right]+\left[1^{2}\right]$	$D_5\left(a_1\right)$
A_2	A_2	$[3]$	$\left[1^{3}\right]$	$E_6\left(a_3\right)$
$3A_1$	$3A_1$	$[2]+[2]+[2]$	$\left[1^{2}\right]+\left[1^{2}\right]+\left[1^{2}\right]$	$E_6\left(a_3\right)$
$2A_1$	$2A_1$	$[2]+[2]$	$\left[1^{2}\right]+\left[1^{2}\right]$	D_5
A_1	A_1	$[2]$	$\left[1^{2}\right]$	$E_6\left(a_1\right)$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	E_8

E_7

$\mathcal{O}$	$\mathfrak{l}$	$\mathcal{O}_{\mathfrak{l}}$	$\mathcal{O}_{\mathfrak{l}}^{\vee}$	$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}^{\vee})$
E_7	E_7	E_7	$\mathbf{0}$	$\mathbf{0}$
$E_7(a_1)$	E_7	$E_7(a_1)$	A_1	A_1
$E_7(a_2)$	E_7	$E_7(a_2)$	$2A_1$	$2A_1$
$E_7(a_3)$	E_7	$E_7(a_3)$	A_2	A_2
D_6	D_6	$[11,1]$	$[1^{12}]$	A_2
E_6	E_6	E_6	$\mathbf{0}$	$(3A_1)''$
$E_6(a_1)$	E_6	$E_6(a_1)$	A_1	A_2+A_1
$E_7(a_4)$	E_7	$E_7(a_4)$	A_2+2A_1	A_2+2A_1
$D_6(a_1)$	D_6	$[9,3]$	$[2^2,1^8]$	A_3
D_5+A_1	D_5+A_1	$[9,1]+[2]$	$[1^{10}]+[1^2]$	$2A_2$
A_6	A_6	$[7]$	$[1^7]$	A_2+3A_1
$E_7(a_5)$	E_7	$E_7(a_5)$	$D_4(a_1)$	$D_4(a_1)$
$D_6(a_2)$	D_6	$[7,5]$	$[2^4,1^4]$	$D_4(a_1)$
A_5+A_1	A_5+A_1	$[6]+[2]$	$[1^6]+[1^2]$	$D_4(a_1)$
D_5	D_5	$[9,1]$	$[1^{10}]$	$(A_3+A_1)''$
$E_6(a_3)$	E_6	$E_6(a_3)$	$A_2 = ind_{A_5}^{E_6}([1^6])$	$D_4(a_1)+A_1$
$(A_5)'$	$(A_5)'$	$[6]$	$[1^6]$	$D_4(a_1)+A_1$
$D_5(a_1)+A_1$	D_5+A_1	$[9,1]+[2]$	$[1^{10}]+[1^2]$	A_3+A_2
A_4+A_2	A_4+A_2	$[5]+[3]$	$[1^5]+[1^3]$	$A_3+A_2+A_1$
$D_5(a_1)$	D_5	$[7,3]$	$[2^2,1^6]$	A_4
D_4+A_1	D_4+A_1	$[7,1]+[2]$	$[1^8]+[1^2]$	A_4
A_4+A_1	A_4+A_1	$[5]+[2]$	$[1^5]+[1^2]$	A_4+A_1
$(A_5)''$	$(A_5)''$	$[6]$	$[1^6]$	D_4
$A_3+A_2+A_1$	$A_3+A_2+A_1$	$[4]+[3]+[2]$	$[1^4]+[1^3]+[1^2]$	A_4+A_2
A_4	A_4	$[5]$	$[1^5]$	$D_5(a_1)$
A_3+A_2	A_3+A_2	$[4]+[3]$	$[1^4]+[1^3]$	$D_5(a_1)+A_1$
$D_4(a_1)+A_1$	D_4+A_1	$[5,3]+[2]$	$[2^2,1^4]+[1^2]$	$(A_5+A_1)'$
A_3+2A_1	A_3+2A_1	$[4]+[2]+[2]$	$[1^4]+[1^2]+[1^2]$	$(A_5+A_1)'$
D_4	D_4	$[7,1]$	$[1^7]$	D_6
$D_4(a_1)$	D_4	$[5,3]$	$[2^2,1^4]$	A_5+A_2
$(A_3+A_1)'$	$(A_3+A_1)'$	$[4]+[2]$	$[1^4]+[1^2]$	A_5+A_2
$2A_2+A_1$	$2A_2+A_1$	$[3]+[3]+[2]$	$[1^3]+[1^3]+[1^2]$	A_5+A_2
$(A_3+A_1)''$	$(A_3+A_1)''$	$[4]+[2]$	$[1^4]+[1^2]$	D_5
A_2+3A_1	A_2+3A_1	$[3]+[2]+[2]+[2]$	$[1^3]+[1^2]+[1^2]+[1^2]$	A_6
$2A_2$	$2A_2$	$[3]+[3]$	$[1^3+1^3]$	D_5+A_1
A_3	A_3	$[4]$	$[1^4]$	$D_6(a_1)$
A_2+2A_1	A_2+2A_1	$[3]+[2]+[2]$	$[1^3]+[1^2]+[1^2]$	$D_6(a_1)+A_1$
A_2+A_1	A_2+A_1	$[3]+[2]$	$[1^3]+[1^2]$	$E_6(a_1)$
$4A_1$	$4A_1$	$[2]+[2]+[2]+[2]$	$[1^2]+[1^2]+[1^2]+[1^2]$	$E_6(a_1)$
A_2	A_2	$[3]$	$[1^3]$	D_6+A_1
$(3A_1)'$	$(3A_1)'$	$[2]+[2]+[2]$	$[1^2]+[1^2]+[1^2]$	D_6+A_1
$(3A_1)''$	$(3A_1)''$	$[2]+[2]+[2]$	$[1^2]+[1^2]+[1^2]$	E_6
$2A_1$	$2A_1$	$[1^2]+[1^2]$	$[1^2]+[1^2]$	$E_7(a_2)$
A_1	A_1	$[1^2]$	$[1^2]$	$E_7(a_1)$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	E_8

E_8

$\mathcal{O}$	$\mathfrak{l}$	$\mathcal{O}_{\mathfrak{l}}$	$\mathcal{O}_{\mathfrak{l}}^{\vee}$	$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}^{\vee}\right)$
E_8	E_8	E_8	$\mathbf{0}$	$\mathbf{0}$
$E_8(a_1)$	E_8	$E_8(a_1)$	A_1	A_1
$E_8(a_2)$	E_8	$E_8(a_2)$	$2A_1$	$2A_2$
$E_8(a_3)$	E_8	$E_8(a_3)$	A_2	A_2
E_7	E_7	E_7	$\mathbf{0}$	A_2
$E_8(a_4)$	E_8	$E_8(a_4)$	A_2+A_1	A_2+A_1
$E_8(a_5)$	E_8	$E_8(a_5)$	$2A_2$	$2A_2$
D_7	D_7	$[13,1]$	$\left[1^{14}\right]$	$2A_2$
$E_7(a_1)$	E_7	$E_7(a_1)$	A_1	A_3
$E_8(b_5)$	E_8	$E_8(b_5)$	$D_4(a_1)$	$D_4(a_1)$
$E_7(a_2)$	E_7	E_7	$2A_1$	$D_4(a_1)$
E_6+A_1	E_6+A_1	$E_6+[2]$	$\mathbf{0}+\left[1^2\right]$	$D_4(a_1)$
$E_8(a_6)$	E_8	$E_8(a_6)$	$D_4(a_1)+A_1$	$D_4(a_1)+A_1$
$D_7(a_1)$	D_7	$[11,3]$	$\left[2^2,1^{10}\right]$	A_3+A_2
$E_8(b_6)$	E_8	$E_8(b_6)$	$D_4(a_1)+A_2$	$D_4(a_1)+A_2$
A_7	A_7	$[8]$	$\left[1^8\right]$	$D_4(a_1)+A_2$
$E_6(a_1)+A_1$				A_4+A_1
$D_7(a_2)$	D_7	$[9,5]$	$\left[2^4,1^6\right]$	A_4+2A_1
$E_7(a_3)$	E_7	$E_7(a_3)$	A_2	A_4
D_6	D_6	$[11,1]$	$\left[1^{12}\right]$	A_4
E_6	E_6	E_6	$\mathbf{0}$	D_4
D_5+A_2	D_5+A_2	$[9,1]+[3]$	$\left[1^{10}\right]+\left[1^3\right]$	A_4+A_2
$E_6(a_1)$	E_6	$E_6(a_1)$	A_1	$D_5(a_1)$
A_6+A_1	A_6+A_1	$[7]+[2]$	$\left[1^7\right]+\left[1^2\right]$	$A_4+A_2+A_1$
$E_7(a_4)$	E_7	$E_7(a_4)$	A_2+2A_1	$D_5(a_1)+A_1$
$D_6(a_1)$	D_6	$[9,3]$	$\left[2^2,1^8\right]$	$E_6(a_3)$
D_5+A_1				$E_6(a_3)$
$E_8(a_7)$	E_8	$E_8(a_7)$	$E_8(a_7)$	$E_8(a_7)$
$E_7(a_5)$	E_7	$E_7(a_5)$	$D_4(a_1)$	$E_8(a_7)$
$E_6(a_3)+A_1$	E_6+A_1	$E_6(a_1)+[2]$	$A_1+\left[1^2\right]$	$E_8(a_7)=ind_{E_6+A_1}^{E_8}\left(ind_{A_5+A_1}^{E_6+A_1}(\mathbf{0})\right)$
$D_6(a_2)$	D_6	$[7,5]$	$\left[2^4,1^4\right]$	$E_8(a_7)$
$D_5(a_1)+A_2$	D_5+A_1	$[7,3]+[3]$	$\left[2^2,1^6\right]+\left[1^3\right]$	$E_8(a_7)$
A_5+A_1	A_5+A_1	$[6]+[2]$	$\left[1^6\right]+\left[1^2\right]$	$E_8(a_7)$
A_4+A_3	A_4+A_3	$[5]+[4]$	$\left[1^5\right]+\left[1^4\right]$	$E_8(a_7)$
A_6	A_6	$[7]$	$\left[1^7\right]$	D_4+A_2
D_5	D_5	$[9,1]$	$\left[1^{10}\right]$	D_5
$E_6(a_3)$	E_6	$E_6(a_3)$	A_2	$D_6(a_1)$
A_5	A_5	$[6]$	$\left[1^6\right]$	$D_6(a_1)$
D_4+A_2	D_4+A_2	$[7,1]+[3]$	$\left[1^8\right]+\left[1^3\right]$	A_6
$A_4+A_2+A_1$	$A_4+A_2+A_1$	$[5]+[3]+[2]$	$\left[1^5\right]+\left[1^3\right]+\left[1^2\right]$	A_6+A_1
$D_5(a_1)+A_1$	D_5+A_1	$[7,3]+[2]$	$\left[2^4,1^6\right]+\left[1^2\right]$	$E_7(a_4)$
A_4+A_2	A_4+A_2	$[5]+[3]$	$\left[1^5\right]+\left[1^3\right]$	D_5+A_2
A_4+2A_1	A_4+2A_1	$[5]+[2]+[2]$	$\left[1^5\right]+\left[1^2\right]+\left[1^2\right]$	$D_7(a_2)$
$2A_3$	$2A_3$	$[4]+[4]$	$\left[1^4\right]+\left[1^4\right]$	$D_7(a_2)$
$D_5(a_1)$	D_5	$[7,3]$	$\left[2^2,1^6\right]$	$E_6(a_1)$
D_4+A_1	D_4+A_1	$[7,1]+[2]$	$\left[1^8\right]+\left[1^2\right]$	$E_6(a_1)$
A_4+A_1	A_4+A_1	$[5]+[2]$	$\left[1^5\right]+\left[1^2\right]$	$E_6(a_1)+A_1$
$D_4(a_1)+A_2$	D_4+A_2	$[5,3]+[3]$	$\left[2^2,1^4\right]+\left[1^3\right]$	$E_8(b_6)$
$A_3+A_2+A_1$	$A_3+A_2+A_1$	$[4]+[3]+[2]$	$\left[1^4\right]+\left[1^3\right]+\left[1^2\right]$	$E_8(b_6)$
A_4	A_4	$[5]$	$\left[1^5\right]$	$E_7(a_3)$
A_3+A_2	A_3+A_2	$[4]+[3]$	$\left[1^4\right]+\left[1^3\right]$	$D_7(a_1)$
$D_4(a_1)+A_1$	D_4+A_1	$[5,3]+[2]$	$\left[2^2,1^4\right]+\left[1^2\right]$	$E_8(a_6)$
A_3+2A_1	A_3+2A_1	$[4]+[2]+[2]$	$\left[1^4\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(a_6)$
$2A_2+2A_1$	$2A_2+2A_1$	$[3]+[3]+[2]+[2]$	$\left[1^3\right]+\left[1^3\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(a_6)$
D_4	D_4	$[7,1]$	$\left[1^8\right]$	E_6
$D_4(a_1)$	D_4	$[5,3]$	$\left[2^2,1^4\right]$	$E_8(b_5)$
A_3+A_1	A_3+A_1	$[4]+[2]$	$\left[1^4\right]+\left[1^2\right]$	$E_8(b_5)$
$2A_2+A_1$	$2A_2+A_1$	$[3]+[3]+[2]$	$\left[1^3\right]+\left[1^3\right]+\left[1^2\right]$	$E_8(b_5)$
$2A_2$	$2A_2$	$[3]+[3]$	$\left[1^3\right]+\left[1^3\right]$	$E_8(a_5)$
A_2+3A_1	$A_2+3\vec{A}1$	$[3]+[2]+[2]+[2]$	$\left[1^3\right]+\left[1^2\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(a_5)$
A_3	A_3	$[4]$	$\left[1^4\right]$	$E_7(a_1)$
A_2+2A_1	A_2+2A_1	$[3]+[2]+[2]$	$\left[1^3\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(b_4)$
A_2+A_1	A_2+A_1	$[3]+[2]$	$\left[1^3\right]+\left[1^2\right]$	$E_8(a_4)$
$4A_1$	$4A_1$	$[2]+[2]+[2]+[2]$	$\left[1^2\right]+\left[1^2\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(a_4)$
$2A_1$	$2A_1$	$[2]+[2]$	$\left[1^2\right]+\left[1^2\right]$	$E_8(a_2)$
A_2	A_2	$[3]$	$\left[1^3\right]$	$E_8(a_3)$
$3A_1$	$3A_1$	$[2]+[2]+[2]$	$\left[1^2\right]+\left[1^2\right]+\left[1^2\right]$	$E_8(a_3)$
A_1	A_1	$[2]$	$\left[1^2\right]$	$E_8(a_1)$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	E_8