Whittaker Vectors,
Generalized Hypergeometric Functions,
and a Matrix Calculus

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1. Representation Theoretical Motivation: smooth Whittaker vectors and orbital invariants of irreducible representations of semisimple Lie groups

Let G be a non-compact, connected, real, reductive Lie group with Cartan involution θ, and and let $\mathfrak{g}$ be the Lie algebra of G with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{k}$ being the Lie algebra of the maximal compact subgroup K corresponding to θ. For a nilpotent element $X \in \mathfrak{g}$, by the Jacobson-Morozov theorem, there exists a standard $\mathfrak{sl}_2$-triple $\{X, H, Y\} \subset \mathfrak{g}$, and a corresponding decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \qquad ; \qquad \mathfrak{g}(j) = \{Z \in \mathfrak{g} \mid ad(H) Z = jZ\}$$

into eigenspaces of the semisimple element H. Let $\mathfrak{n}$, $\overline{\mathfrak{n}}$ be the nilpotent subalgebra spanned by, respectively, the positive and negative eigenspaces of H in $\mathfrak{g}$, and let $\mathfrak{l} = \mathfrak{g}(0)$. Let χ_X denote the linear form on $\mathfrak{g}$ defined by

$$\chi_X(Z) = B(Y, Z)$$

where $B(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$. Then there is an ideal $\mathfrak{n}_X$ in $\mathfrak{n}$ such that $\chi_X([\mathfrak{n}_X, \mathfrak{n}_X]) = 0$, and $\dim(\mathfrak{n}/\mathfrak{n}_X) = \frac{1}{2}\dim \mathfrak{g}(1)$. Now let (π, V) be a continuous admissible representation of G in a Hilbert space V, let V^∞ be the subspace of smooth vectors and let $V^{-\infty}$ be the continuous dual (with respect to the usual Frèchet topology) of V^∞. The space of smooth Whittaker vectors for the representation (π, V) corresponding to the nilpotent element $X \in \mathfrak{g}$ is

$$Wh_X^\infty(\pi) = \left\{T \in V^{-\infty} \mid Z \cdot T = i\chi(Z)\, T \quad , \quad \forall\ Z \in \mathfrak{n}_X\right\}$$

A fundamental idea developed over the twenty-seven years since Kostant's original paper ([Ko1]) is that the dimension of this space should be non-zero and finite precisely when the nilpotent element X lies in the wave front set ([Ro], [Vo2]) of π. In fact, a stronger conjecture ([Ma], [NOTYK]), is that this dimension should be related to the multiplicity of the orbit $G \cdot X$ in the wave-front cycle [BV] of π, and then, by the Vogan conjecture (proved by Schmid and Vilonen [SV]) to the multiplicity of corresponding nilpotent K_C-orbit in the characteristic cycle of π. Indeed, one can attach to a representation π, a *smooth Whittaker cycle*, i.e., a formal sum of the form

$$wc(\pi) = \sum_{\mathcal{O} \subset \mathcal{N}} \dim\left(Wh_{X_{\mathcal{O}}}^\infty(\pi)\right) \cdot \overline{\mathcal{O}} \quad ,$$

where $\mathcal{N}$ is the set of nilpotent G-orbits and $X_{\mathcal{O}}$ is a representative element of a G-orbit $\mathcal{O}$ in $\mathcal{N}$, and then cast the conjectured relationship between dimensions of spaces of smooth Whittaker vectors and multiplicities of orbits in the characteristic cycle as an extension of the Vogan conjecture.

As an initial tack to uncovering the correspondence between characteristic cycles and Whittaker cycles, we have explored the details of the correspondence for a family of small unitary representations associated with simple non-euclidean Jordan algebras (see [Sa], [KS], [BSZ]). This happens to be a conveniently simple case, for each of these representations, the support of the characteristic cycle is the closure of a single $K_{\mathbb{C}}$ orbit, and the multiplicity of this orbit in the characteristic cycle is one. During the course of this investigation, we also computed the Bernstein degrees [Vo1] of these representations; and this computation has rather interesting connections with generalized Selberg integrals and generalized hypergeometric functions that we intend to explore further.

The unipotent representations studied arise as follows: let G be a simple Lie group, K a maximal compact subgroup corresponding to a Cartan involution θ. Assume that G has a parabolic subgroup P with Levi decomposition $P = LN$, such that N is abelian and P is conjugate to $\theta(P)$. In such a situation, the nilpotent Lie algebra $\mathfrak{n} = Lie\,(N)$ has a natural Jordan algebra structure, L is the group of automorphisms of $\mathfrak{n}$, and G is the "conformal group" of $\mathfrak{n}$. Write $M = L \cap K$, let $\mathfrak{t}$ be a maximal toral subalgebra in the orthogonal complement of $\mathfrak{m} = Lie\,(M)$ in $\mathfrak{k} = Lie\,(K)$, and let $\Sigma = \Sigma\,(\mathfrak{k};\mathfrak{t})$ be the restricted root system for $\mathfrak{t}$ in $\mathfrak{k}$. It turns out that are only three possibilities for Σ: $\Sigma = A_{n-1}$, C_n, or D_n, where $n = \dim(\mathfrak{t})$. Moreover, there is a uniform prescription for writing down the restricted root systems Σ: one can adopt a Euclidean basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\mathfrak{t}^*$ such that simple roots of $\mathfrak{t}$ in $\mathfrak{k}$ take the form $\{\gamma_i - \gamma_{i-1}\}$, $i = 1, \ldots, n-1$, together with $2\gamma_n$ and $(\gamma_{n-1} + \gamma_n)$ in the cases where, respectively, $\Sigma = C_n$ and D_n. It then happens that, for a given G, the short roots $\pm\gamma_i \pm \gamma_j \in \Sigma$ all have a common multiplicity, d, and the long roots $2\gamma_i$ all have a common multiplicity e (and both of the integers d and e can be related back the Jordan algebra structure of $\mathfrak{n}$). Set $r = d(n-1)+e$ and let ν be the positive character for L such that v^{2r} is the determinant of the adjoint action of L on $\mathfrak{n}$. For $t \in \mathbb{R}$, let $I\,(t)$ denote the (normalized) induced representation $Ind_P^G\,(\nu^t)$.

2. Kostant's Construction

In [K2], B. Kostant provides an explicit construction of a family π_r, $r \in (-1, \infty)$, of unitary irreducible representations of simply connected covering group of $SL\,(2, \mathbb{R})$ in terms of L_2 functions on $(0, \infty) \subset \mathbb{R}$, and determines the smooth Whittaker vectors for the nilpotent elements $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{sl}_2$. Let $\mathfrak{n} = span\,(f)$ and $\overline{\mathfrak{n}} = span\,(e)$. (We remark, that although it is more conventional to regard $\mathfrak{n}, \overline{\mathfrak{n}}$ as, respectively, the subalgebras of strictly upper and strictly lower triangular matrices In what follows, we shall be considering realizations obtained by restricting the corresponding principal series representation to $\overline{N} \sim \overline{\mathfrak{n}}$ and then carrying out a Fourier transform.) In this realization, for each $X \in (0,\infty)$, there is a unitary character $\chi_X = i\, Tr\,(X\cdot)$ of $\overline{\mathfrak{n}}$ and Kostant shows that the corresponding delta functional $\delta_X \in L_2\,(0,\infty)^{-\infty}$ provides a basis for the space of smooth Whittaker vectors $Wh^{\infty}_{\overline{n}}\left(L_2\,(0,\infty)^{-\infty}\right)$. Kostant also shows that, in this realization, the smooth Whittaker vectors corresponding to the opposite nilpotent subalgebra $\mathfrak{n}$ correspond to integration against a certain (modified) Bessel functions, and that the two kinds of smooth Whittaker functionals are related via a Hankel transform.

There are several salient features of Kostant's construction that we hope to replicate for a particular family of unipotent representations of $GL\,(2n,\mathbb{R})$; namely, that:

- the representations are realized in a L_2-space; in our case the L_2 functions on an L-orbit $\mathcal{O}$ in $\mathfrak{n}$;
- one can realize the smooth Whittaker vectors for one nilpotent subalgebra $\overline{\mathfrak{n}}$ as delta functionals δ_X, $X \in \mathcal{O}$;
- the differential equations for smooth Whittaker functions for the opposite nilpotent $\mathfrak{n}$ are of hypergeometric type
- the asymptotics of solutions of these differential equations as their argument x approaches the algebraic-geometric boundary of the orbit and as x approaches ∞ allow one to prove that integrating against these "Whittaker functions" yields a continuous linear functional on the space of smooth vectors;
- the action of the long Weyl group element which provides a mapping from Whittaker vectors for $\mathfrak{n}$ to Whittaker vectors for $\overline{\mathfrak{n}}$, can be implemented by a generalized Hankel transform.

The representations of $GL\,(2n,\mathbb{R})$ we are considering are singular unitary representations π_q, that occur as the unique irreducible unitarizable quotients of normalized principle series representations $Ind_P^G\,(\nu_q)$ $(q = 1,\ldots,n)$, of the type studied in the preceding project. By restricting to $\overline{N} \approx \overline{\mathfrak{n}}$ we get the "non-compact picture" of $Ind_P^G\,(v_q)$ and, by Fourier transform ([BSZ]), a realization of the unitarizable quotient in terms in $\mathcal{H}_q = L^2\,(\mathcal{O}_q, d\mu_q)$, $\mathcal{O}_q = L\cdot X_q$ being a particular orbit[1] of the Levi subgroup $L = MA$ of P in $\mathfrak{n}$ that comes equipped with a uniquely defined equivariant measure $d\mu_q$. Fourier transforming the canonical action of $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ in the non-compact picture, we obtain partial differential equations for "Whittaker distributions" on $\mathcal{O}_q$. As in Kostant's study of $\widetilde{SL\,(2,\mathbb{R})}$, it turns out [BZ] that the smooth Whittaker functionals corresponding to the nilpotent subalgebra $\overline{\mathfrak{n}}$ are multiples of Dirac distributions δ_y, $y \in \mathcal{O}_q$ on $L_2\,(\mathcal{O}_q, d\mu_q)$.

[1]These orbits are the L-orbits of the $X_q = c_1 + \cdots + c_q$, where the $\{c_i\}_{i=1}^n$ is a set of primitive idempotents of $\mathfrak{n} \approx M_{n,n}\,(\mathbb{R})$ regarded as a Jordan algebra.

The ultimate goal of this project is to show that the smooth Whittaker functionals corresponding to the opposite nilpotent $\mathfrak{n}$ correspond to integration against certain generalized hypergeometric functions, in a manner analogous to Kostant's use of Bessel functions. To do so, however, we not only need to exhibit solutions ψ of the Whittaker PDEs, but also to demonstrate that they decay rapidly enough at ∞ and upon approach to the (algebraic-geometric) boundary of the orbit so that

$$\int_{\mathcal{O}_q} \psi f d\mu_q < \infty$$

for any smooth vector $f \in L_2\left(\mathcal{O}_q, d\mu_q\right)$.

Now it turns out the that, in terms of the natural Euclidean coordinates for $\mathfrak{n} \simeq M_{n,n}\left(\mathbb{R}\right) \simeq \mathbb{R}^{n^2}$, the Whittaker condition

$$Z \cdot \psi\left(\mathbf{x}\right) = i \left\langle Y, Z\right\rangle \psi\left(\mathbf{x}\right) \qquad , \quad \forall\ Z \in \mathfrak{n}_q$$

reduces to the problem of finding conjugacy invariant solutions of the following system of PDEs for a function ψ on $M_{q,q}\left(\mathbb{R}\right)$ (the space of $q \times q$ real matrices)

$$\left(\sum_{k=1}^{q} x_{ik} \frac{\partial}{\partial x_{ki}} \frac{\partial}{\partial x_{jk}} - s \frac{\partial}{\partial x_{ij}} - \lambda \delta_{ij}\right) \psi = 0 \qquad , \quad i, j = 1, \ldots, q \tag{4}$$

This brings us to

Objective 2.1. Demonstrate that the differential equations (4) have generalized hypergeometric functions as their unique, analytic, conjugacy invariant solutions.

We note that hypergeometric functions of matrix arguments have been studied for many years ([He], [Ja] [Mu]) and that they remain a very prominent topic in the mathematical literature. ([GR], [Ko], [VK], [HO]). To make explicit contact with the literature, however, is not exactly straightforward. For the explicit expressions for the hypergeometric functions of matrix argument that occur in the literature are typically expressed in terms of the eigenvalues of matrices lying in a symmetric cone rather than the matrix arguments themselves. Moreover, such hypergeometric functions that are typically written down as infinite series of the form

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; \mathbf{x}\right) = \sum_{d=0}^{\infty} \sum_{\lambda \in \mathcal{P}(d)} \frac{\left[a_1\right]_\lambda \cdots \left[a_p\right]_\lambda}{\left[b_1\right]_\lambda \cdots \left[b_q\right]_\lambda d!} \Phi_\lambda\left(\mathbf{x}\right)$$

where $\mathcal{P}\left(d\right)$ is the set of partitions of d, $\left\{\Phi_\lambda | \lambda \in \mathcal{P}\right\}$ as is a basis for the symmetric polynomials, and $\left[a\right]_\lambda$ is a generalized Pochhammer symbol corresponding to the partition λ (the precise form of which depends on the choice of $\left\{\Phi_\lambda\right\}$). We remark that the choice of basis $\left\{\Phi_\lambda\right\}$ is not meant to seem arbitrary. Initially, only zonal polynomials were used. Even today, many of the properties of such generalized hypergeometric functions, for example, that they satisfy certain differential equations, are proved by appealing to the special properties, especially combinatorial properties, of the bases $\left\{\Phi_\lambda\right\}$.

3. The Matrix Calculus

We have developed a matrix calculus that is not only well-suited for connecting our generalized Whittaker PDEs with generalized hypergeometric functions, it also suggests a very natural, uniform origin for the holonomic systems of PDEs ([Kas], [GG]) associated generalized hypergeometric functions of matrix argument ([Ja], [GR], [Ko], [Ki]). Let $\mathbf{X} = (x_{ij})_{1\leq i,j\leq n}$ be an $n \times n$ matrix of indeterminates. The characteristic polynomial of $\mathbf{X}$ is

$$\det(\mathbf{X} - \lambda\mathbf{I}) = (-1)^n \lambda^n + p_1(x)\lambda^{n-1} + \cdots + p_n(x)$$

where of course $p_1(\mathbf{X}) = tr(\mathbf{X})$, $p_n(\mathbf{X}) = \det(\mathbf{X})$ and the intermediary p_i are the so-called generalized determinants. In what follows, it is convenient to replace the p_i by

$$\phi_i(\mathbf{X}) = (-1)^{n+1} p_i(\mathbf{X})$$

From classical invariant theory one knows that the p_i, hence the ϕ_i, provide a rational basis for the space of conjugacy invariant polynomials in the entries x_{ij}.

$$\mathbb{C}[x_{ij}]^G \cong \mathbb{C}[\phi_1, \ldots, \phi_n]$$

We also note that in terms of the ϕ_i, the Cayley-Hamilton theorem takes the form

$$\mathbf{X}^n = \phi_1\mathbf{X}^{n-1} + \phi_2\mathbf{X}^{n-2} + \cdots + p_{n-1}\mathbf{X} + p_n\mathbf{I}$$

and that by iterating this identity one obtains formulas of the form

$$\mathbf{X}^{n+q} = \sum_{i=0}^{n-1} \xi_{n,q,i}(\phi_1, \ldots, \phi_n)\mathbf{X}^i$$

which we interprete as allowing us to make a identification

$$\mathbb{C}[\mathbf{X}] \cong span_{\mathbb{C}[x]^G}\left[\mathbf{I}, \mathbf{X}, \ldots, \mathbf{X}^{n-1}\right]$$

Next we introduce the operator

$$\mathbf{D} = \left(\frac{\partial}{\partial x_{ji}}\right)_{\substack{i=1,\ldots n\\ j=1,\ldots,n}}$$

which acts naturally on $\mathbb{C}[\mathbf{X}] \cong span_{\mathbb{C}[x]^G}[\mathbf{I}, \mathbf{X}, \ldots, \mathbf{X}^{n-1}]$ via the rule

$$(\mathbf{D\Phi})_{ij} = \sum_{k=1}^{n} (\mathbf{D})_{ik} (\Phi)_{kj} = \sum_{k=1}^{n} \frac{\partial}{\partial x_{ki}} (\mathbf{\Phi})_{kj}$$

as well as on conjugacy invariant polynomials via "scalar multiplication from the left". In fact, one finds

$$\mathbf{D}\phi_q = \mathbf{X}^{q-1} - \phi_1\mathbf{X}^{q-2} - \cdots - \phi_{q-1}\mathbf{I}$$

Remark 3.1. It is easy to also see that $\mathbf{D}$ obeys a Liebnitz-like rule when acting on $\mathbb{C}[x]^G$

$$\mathbf{D}(fg) = (\mathbf{D}f)g + f(\mathbf{Dg})$$

and that

$$\mathbf{DX} = n\mathbf{I}$$

However,

$$\mathbf{DX} - \mathbf{XD} \neq \mathbf{I}$$

It's not even close. Thus, this is not some kind of generalized Heisenberg algebra.

Actually, it's the "Euler operator" $\mathbf{XD}$ or more explicitly

$$(\mathbf{XD})_{ij} = \sum_k x_{ik} \frac{\partial}{\partial x_{jk}}$$

that's of most utility for us. One has

$$\mathbf{XD}\left(\phi_q\right) = \mathbf{X}^q - \sum_{i=1}^{q-1} \phi_i \mathbf{X}^{q-i} \tag{5}$$

$$\mathbf{XD}\left(\mathbf{X}^q\right) = n\mathbf{X}^q + \sum_{i=1}^{q-1} \psi_i \mathbf{X}^{q-i}$$

Here the functions ψ_i, $i = 1, \ldots, n$ are defined by

$$\psi_i\left(\mathbf{X}\right) = tr\left(\mathbf{X}^n\right)$$

and are related to the generalized determinants ϕ_i via

$$\psi_i\left(\mathbf{X}\right) = \det \begin{bmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ -2\phi_2 & \phi_1 & & & 0 \\ 3\phi_3 & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ (-1)^{n+1} q\phi_q & (-1)^q \phi_{q-1} & \cdots & \cdots & \phi_1 \end{bmatrix} \tag{6}$$

To the erudite this should look a bit like the formula relating power symmetic functions to elementary symmetric functions that goes back to Newton. Indeed, that's a good hint as to how (6) is proved.

Okay, now for the first punch line that you must have seen coming, but hopefully not so simply put:

Fact 3.2. *In terms of the operators* $\mathbf{X}$ *and* $\mathbf{D}$*, the system (4) has a particularly simple expression:*

$$\left(\mathbf{XDXD} - s\mathbf{XD} - \lambda\mathbf{X}\right)\psi = 0 \quad . \tag{6}$$

We note that when $n = 1$, this matrix differential equation reduces to the ordinary differential equation for the confluent hypergeometric functions (see, e.g, [AAR], pg. 188). Secondly, noting the prominence of the "Euler operator" $\mathbf{E} = \mathbf{XD}$, it is natural to view (6) as the special case (the confluent case) of a matrix calculus equation for a hypergeometric function ${}_pF_q$; by which we mean a equation of the form

$$\left[\mathbf{E}\left(\mathbf{E} - b_1\right)\cdots\left(\mathbf{E} - b_q\right) - \mathbf{X}\left(\mathbf{E} + a_1\right)\cdots\left(\mathbf{E} + a_p\right)\right] F = 0 \quad .$$

But even more striking than these notational niceties is the fact that identities such as (5) enable one to actually solve (6) via a generalized power series technique. Explicitly, one makes the general ansatz

$$\Psi = \sum_{m \in \mathbb{N}^n} a_{m_1,\ldots,m_n} \psi_1^{m_1+r_1} \cdots \psi_n^{m_n+r_n} \tag{7}$$

and discovers that for non-integer s there is a unique (up to a constant factor) of conjugacy-invariant solution to (6) without singularities on the locus $\det(\mathbf{X}) = 0$. Indeed, the indicial equations turn out to be $r_i = 0$, $i = 1, \ldots, n-1$, and $r_n(r_n - s) = 0$, and from this we can deduce the behavior of solutions as they approach the boundary of the orbit (where $\psi_n = 0$); determination of this behavior will be used latter in demonstrating that, as an integral kernel, Ψ provides a continuous linear functional on the space of smooth vectors; i.e. a solution Ψ corresponds to a smooth Whittaker vector.

An explicit connection with generalized hypergeometric functions should follow along the following lines. Plugging the ansatz (4) into the matrix calculus version of a generalized hypergeometric equation and using the fact that the resulting total coefficients of $\mathbf{I}, \mathbf{X}, \ldots, \mathbf{X}^{n-1}$ must separately vanish, one can deduce from (6) a system of PDEs with respect to the variables ψ_i. By the conjugacy invariance of the functions ψ_i, the ψ_i actually depend only on the generalized eigenvalues of its arguments. In fact, if one evaluates any of the ψ_i, $i < n$, on a symmetric matrix, one sees that it coincides with the power sum elementary symmetric function $p_i(\lambda_1, \ldots, \lambda_n)$ of its eigenvalues, and that $\psi_n(\mathbf{X}) = \det(\mathbf{X})$ coincides the n^{th} elementary symmetric function $e_n(\lambda_1, \ldots, \lambda_n)$ of its eigenvalues. While it is a difficult problem to explicitly express, for example, the Jack symmetric functions used by [Ko] in terms of the power sums and e_n, it should be possible, as in [Ko], to show that the corresponding holonomic system of PDEs has a unique analytic symmetric solution satisfying $\psi(\mathbf{0}) = 1$, and to display a generalized hypergeometric function as that solution.

In the manner outlined above, verified that, at least for $n = 2$, the regular, conjugacy invariant, solutions of the matrix calculus version of the Gauss hypergeometric equation

$$\begin{aligned} \left[\mathbf{E}(\mathbf{E} - b) - \mathbf{X}(\mathbf{E} + a_1)(\mathbf{E} + a_2)\right]\psi &= 0 \\ \psi(\mathbf{0}) &= 1 \end{aligned}$$

coincides (after adjusting the parameters a_1, a_2 and b) with the generalized Gauss hypergeometric equations for ${}_2F_1$ studied by James [Ja] and Muirhead [Mu].

Our generalized Whittaker functions should correspond to a generalized confluent hypergeometric functions ${}_1F_1$. We thus led to

Objective 2.1′. Show that the matrix calculus equation (6) can be reduced to a holonomic system of second order partial differential equations having

$${}_1F_1(a; b; \mathbf{x}) = \sum_{d=1}^{\infty} \sum_{|\lambda| = d} \frac{[a]_\lambda}{[b]_\lambda \, d!} \Phi_\lambda(\mathbf{x})$$

is the unique analytic symmetric solution satisfying $F(\mathbf{0}) = 1$.

We have in mind here using the Jack symmetric polynomials (and the corresponding generalized Pochhammer symbols) as at least the initial choice for the basis functions Φ_λ; but we remain open to the possibility of a more convenient or natural basis presenting itself. Also, in our intended application, the parameter s will be a positive integer fixed by the choice of the small unitary representation Π_q, and it may be that the Whittaker functions we seek are actually special solutions of the holonomic system of PDEs.

Once the connection with the generalized hypergeometric functions is established, we can then hope to get a handle on asymptotic behavior of the solutions $\Psi(\mathbf{x})$ as $|\mathbf{x}| \to \infty$. In the classical case, this is typically done by appealing to integral representations of hypergeometric functions. Via the integral representations of generalized hypergeometric functions ([GKZ], [Yan]), or by direct analysis of the differential equations [En], we expect to be able to determine the asymptotic behavior of solutions of (6).

Finally, with a handle on the asymptotics of our generalized Whittaker functions at ∞ and upon approach to the singular locus (where $\det(\mathbf{X}) = 0$), we come to the next objective of this project.

Objective 2.2. Show that for the unique K-spherical vector $\zeta_0 \in L_2(\mathcal{O}_q, d\mu_q)$ and the regular, conjugacy invariant solution of Ψ of (2)

$$\langle \Psi, \zeta_0 \rangle = \int_{\mathcal{O}_q} \Psi \zeta_0 d\mu_{\mathcal{O}} \qquad \text{is bounded.} \tag{8}$$

This falls short of our stated goal of demonstrating that such Ψ correspond to smooth $\mathfrak{n}$-Whittaker functionals. However, as the spherical vector corresponds to the "lowest energy state" of the representation, the spherical vector is presumably the state with the most problematic support as $|\mathbf{x}| \to \infty$. Moreover, we expect the K-finite vectors to be non-singular upon approach to the boundary of the orbit, as these K-types correspond to the K-types in the ring of regular functions on the closure of the orbit. Thus, we view demonstrating the validity of (8) as the crux to the problem of proving that $\langle \Psi, \zeta \rangle$ is bounded for any K-finite vector in $L_2(\mathcal{O}_q, d\mu_q)$. Finally, using the fact that the K-finite vectors are dense in $L_2(\mathcal{O}_q, d\mu_q)^\infty$, it would follow that such a Ψ provides, in fact, a smooth Whittaker functional.

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