Lecture 1: Introduction to the Atlas Program

B. Binegar

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, USA

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Problem: Describe the set of equivalences classes of irreducible unitary representations of a general Lie group G.

Why?

- a basis for *abstract harmonic analysis*
- equivalent to a classification of elementary quantum mechanical systems possessing the symmetry group G:
  - in particular, elementary particles - the fundamental building blocks of nature

Major Milestones

- 1930's : case of compact Lie groups solved (Cartan)
- 1950's : initial foray into non-compact groups (Bargmann, Mackey)
- 1960's : nilpotent case solved (Kirillov)
- 1970's : type I solvable case solved (Kostant-Auslander)
- 1980's : case of general algebraic group reduced to that of reductive Lie groups (Duflo)

The Modern Framework

G : real reductive group (the last holdout)

$\widehat{G}_u$: the unitary dual of G, the set of equiv classes of unitary irr reps

$\widehat{G}_{adm}$: the set of equiv classes of irr *admissible* reps

$\widehat{G}_{adm}$ is a "smoother", well understood set containing $\widehat{G}_u$

The unitary dual problem reduces to:

Problem: which of the admissible reps of a reductive group G are unitary?

For a fixed real reductive group G, the enumeration of the admissible reps that are unitary can be formally reduced to a finite computation.

Jeff Adams (circa 2001): Let's compute $\widehat{G}_{adm}$

- Exceptional Lie groups completely
- Classical Lie groups - far enough to develop inductive methods

Plan of Action:

Step 1: Develop a software platform capable of handling the basic structure theory of an arbitrary real reductive group (F. du Cloux $\sim$ 2003).

Step 2: Develop software capable of computing the character theory of the admissible dual of an arbitrary real reductive group. (F du Cloux $\sim$ 2005, E_8 2007)

Step 3: Develop highly efficient algorithms for testing for unitarity (the next white whale)

An Unforeseen Motif:

`atlas` is teaching us representation theory!

Why?

- Developing efficient algorithms requires one to strip a theory down to its bare skeleton. (Da Vinci metaphor)
  (snide remark: fashion accessories are the first to go)
- new tools to probe reductive Lie groups
- data!
  (Brahe-Kepler paradigm)

Outline of Lecture Series

Lecture 1: `atlas` Basics

Lecture 2: `atlas` Examples

Lecture 3: A Combinatorial Parameterization of Nilpotent Orbits

Lecture 4: LS Cells, Twisted Induction and Duality

Lecture 5: Admissible Representations in the Atlas Framework

Lecture 6: Admissible Representations, HC Cells, and Nilpotent Orbits

Lecture 7: Admissible Representations, HC Cells, and Primitive Ideals

Obtaining atlas

Atlas is open source software that is free and readily available from the "downloads" page

`http://atlas.math.umd.edu/software/download`

on the Atlas for Lie Groups and Representations website

`http://atlas.math.umd.edu`

Linux users can have the most recent release of atlas up and running in almost no time at all. Simply download the file

`http://atlas.math.umd.edu/software/download/atlas_0.3.exe`

into a suitable directory (e.g $HOME/atlas) and execute atlas_0.3.exe from that directory.

Windows users can also download an "executable" version for their platform:

`http://atlas.math.umd.edu/software/download/atlas4win_0.3.zip`

Users of other operating systems (e.g. Mac users, Sun users) can download and extract the source code tarball

`http://atlas.math.umd.edu/software/download/atlas_0.3.tgz`

and then compile the source code themselves.

(If you have a recent version of `g++` as well as the development versions of the `readline` and `ncurses` libraries this compilation process is also simple as entering the `make` command in the source code directory.)

Starting and Exiting atlas

Once atlas has been downloaded and installed, a user can startup atlas and be greeted with the following message:

```
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: May  11 2008 at 09:04:58.
Enter ''help'' if you need assistance.

empty: qq
```

Before re-entering atlas, it will be most helpful to first have a general idea of atlas's various *modes* of operation.

atlas modes are certain command environments which are attuned to the level of specificity of a particular underlying representation theoretical context.

There are five modes:

mode	context	choices made	specificity
empty	*none*	*none*	none
help	*none*	*none*	
main	Lie algebra, cpx red group G	roots,coroots, (inner class)	low
real	real reductive Lie group $G_{\mathbb{R}}$	above plus a real form	$\updownarrow$
block	block of irr adm reps	above plus a dual real form	highest

Entering a "higher" mode command in a "lower" mode automatically takes you to that mode (you will be prompted for the additional data needed to initialize the higher mode).

Entering q will let you descend to back to a lower mode.

When `atlas` starts it immediately enters the `empty` mode. This is indicated by the `empty:` prompt.

In the `empty` mode no data structures have been initialized and `atlas` is simply waiting to be told what do.

The only commands that are available in this mode are

`help` : enter help mode

`type` : used to select the Lie algebra, isogeny class, and inner class of the group to be studied.

Entering `help` moves you to `help` mode.

In `help` mode, you can view a list of available commands by entering "?".

Or, enter a command name to get a brief explanation of what a command does and how its output is formatted.

Enter "q" to exit `help` mode

After entering the type command and entering the data necessary to specify the inner class of real forms one arrive in the main mode (this will be indicated by the main: prompt).

At this point atlas has already done some computations is more than happy to answer questions about root systems and the possible real forms.

A brief listing of the commands that are available in the main mode:

- `cmatrix` : prints the Cartan matrix
- `roots` : outputs the roots in the lattice basis
- `posroots` : outputs the positive roots in the lattice basis
- `coroots` : outputs the coroots in the lattice basis
- `poscoroots` :
- `simpleroots` : outputs the simple roots in the lattice basis
- `simplecoroots` :

- `roots_rootbasis` : outputs the roots in terms of a basis of simple roots
- `posroots_rootbasis`: basis of simple roots
- `coroots_rootbasis` : outputs the coroots in terms of a basis of simple roots
- `poscoroots_rootbasis` : outputs the positive coroots in terms of a basis of simple roots
- `showrealforms` : outputs the (weak) real forms in the inner class
- `showdualforms` : outputs the (weak) real forms in the dual group
- `blocksizes` : outputs the matrix of blocksizes
- `realform` : sets the real form for the group and moves you to `real` mode

Once one enters the `real` mode the following commands are available:

- `cartan` : lists the conjugacy classes of Cartan subgroups of the real form and their associated characteristics
- `components` : describes the component group of the real group
- `realweyl` : outputs the structure of the real Weyl group (for a specified class of Cartan subsubgroups)
- `kgb` : outputs the orbits of K on G/B
- `dualkgb` : outputs the KGB data for a dual real form
- `gradings` : outputs information about the grading of the imaginary root system (for a specified class of Cartan subgroups)
- `strongreal` : outputs information about the strong real forms

block mode

The block mode corresponds to a further specialization on the atlas environment in which in addition to the specification of a real group, a real form of dual complex group is also specified.

This is the mode in which information gleamed from KLV (Kahzdan-Lusztig-Vogan) computations can be obtained.

One enters block mode (e.g. from real mode) by entering any one of the following commands

- block : lists all the representations in a block and some of their invariants
- blockd : similar to block command with alternative output format
- blockorder : prints the covering relations of the Bruhat order on a block
- blockstablizer : prints the real Weyl group of a block
- blocku : prints the unitary representations in the block at infinitesimal character ρ

- `blockwrite` : writes the block information to a file
- `extract-cells` : reads the cell and KLV binary files and prints the W-cells information
- `extract-graph` : reads the cell and KLV binary files and prints the W-graph information
- `klwrite` : writes the KLV polynomials to a disk in binary format
- `klbasis` : prints the KLV basis for the Hecke module
- `kllist` : prints the list of distinct KLV polynomials
- `wcells` : prints the KLV cells for the block
- `wgraph` : prints the W-graph for the block

Basic Recipe:

Step 1 : Start with a product of simple, simply connected, complex groups and complex torii.

$$\widetilde{G} = G_1 \times G_2 \times \cdots \times G_k \times (\mathbb{C}^\times)^\ell$$

Step 2: mod out by a finite subgroup Γ of the center of $\widetilde{G}$

Step 3: choose a real form of $\widetilde{G}/\Gamma$

Examples:

- $SL(2,\mathbb{C}) \to SL(2,\mathbb{C}/\{I,-I\} \sim PSL(2,\mathbb{C}) \to SO(2,1)$
- $SL(2,\mathbb{C}) \times \mathbb{C}^\times \to SL(2,\mathbb{C}) \times \mathbb{C}^\times / \{(I,1),(-I,-1)\} \to GL(2,\mathbb{R})$

Step 1: Specification of Lie type

Step 2: Specification of isogeny type

Step 3: Selection of an inner class

Step 4: Selection of a real form

Step 1: Specification of Lie type

```
This is the Atlas of Reductive Lie Groups Software Package version 0.2.6.2.
Build date: May 18 2007 at 15:46:34.
Enter ''help'' if you need assistance.

empty:
```

```
This is the Atlas of Reductive Lie Groups Software Package version 0.2.6.2.
Build date: May 18 2007 at 15:46:34.
Enter ''help'' if you need assistance.
empty: type
Lie type:
```

At this point `atlas` is asking for the Cartan type of the complexified Lie algebra $\mathfrak{g}$ of the group to be constructed.

The user should enter a sequence of expressions of the form $\mathtt{Xn_i}$ separated by periods "."

Each "term" $\mathtt{Xn_i}$ indicates $\mathfrak{g}$ has a simple factor of Cartan type X and rank $\mathtt{n_i}$;

Well, except for torus factors which are indicated by expressions of the form $\mathtt{Tn_i}$.

Example:

To constuct a group G such that

$$Lie(G) = \mathfrak{f}_4 \oplus \mathfrak{so}\,(13) \oplus \mathbb{C} \oplus \mathfrak{sp}\,(4) \oplus \mathfrak{so}\,(8)$$

Enter

```
F4.B6.T1.C2.D4
```

at the `type` prompt.

The first step tells `atlas` the simply connected covering group $\widetilde{G}$ of the complex linear group G to be constructed.

The next step is to specify for `atlas` a finite subgroup Γ of the center of $\widetilde{G}$ such that

$$G = \widetilde{G}/\Gamma$$

Suppose you have just entered `A4.T1.T1`

Immediately after entering your choice, `atlas` will say something like

```
elements of finite order in the center of the simply connected group:
Z/5.Q/Z.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
```

Explanation

The desired group G must be a quotient of the simply connected group $\widetilde{G}$ by some finite subgroup Γ of the center $\widetilde{Z}$ of $\widetilde{G}$.

To specify the subgroup Γ you must give `atlas` a list of its generators.

This you must do (barring any short cuts to be mentioned below) by specifying the generators of the "components" of Γ lying in each simple factor of $\widetilde{G}$.

G	Γ_{adj}	possible generators
A_n	$\mathbb{Z}_{n+1}$	$\frac{0}{n+1}, \frac{1}{n+1}, \ldots, \frac{n}{n+1}$
B_n	$\mathbb{Z}_2$	$\frac{0}{2}, \frac{1}{2}$
C_n	Z_2	$\frac{0}{2}, \frac{1}{2}$
D_n, n odd	$\mathbb{Z}_2$	$\frac{0}{2}, \frac{1}{2}$
D_n, n even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\left\{\frac{0}{2}, \frac{1}{2}\right\} \times \left\{\frac{0}{2}, \frac{1}{2}\right\}$
E_6	$\mathbb{Z}_3$	$\frac{0}{3}, \frac{1}{3}, \frac{2}{3}$
E_7	$\mathbb{Z}_2$	$\frac{0}{2}, \frac{1}{2}$
E_8	1	
F_4	1	
G_2	1	

($\mathbb{Z}_n \equiv \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n.)

When $\widetilde{G}$ has a factor of type T_1, one must specify the generators of a finite subgroup of $\mathbb{C}^\times$.

$$\{\text{elements of finite order in } \mathbb{C}^\times\} \sim \mathbb{Q}/\mathbb{Z}$$

$\longrightarrow$ generators of $\Gamma \subset T \quad \sim p\backslash q \in \mathbb{Q}$

$\longrightarrow$ Γ is specified by giving `atlas` a list of fractions; e.g. 1/3,1/5,1/8

Shortcuts:

- to utilize the simply connected group, simply hit RETURN at the prompt for kernel generators (or enter `sc`)
- to utilize the adjoint group, simply enter `ad` at the prompt for kernel generators.

Digression: Two methods of specifying a real form:

- method 1:
  - specify an anti-holomorphic involution γ of G and set
$$G_{\mathbb{R}} = G^{\gamma} \equiv \text{set of } \gamma \text{ fixed points}$$
- method 2:
  - fix a anti-holomorphic involution γ_o corresponding to the compact real form of G
  - specify a **holomorphic** involution θ such that $\theta\gamma_o = \gamma_o\theta$.
  - Set
$$G_{\mathbb{R}} = G^{\theta\gamma_o}$$

`atlas` uses the second method

complex group $K = G^{\theta}$ is particularly important

Consider the following exact sequence of group homomorphisms

$$1 \rightarrow Int\,(G) \longrightarrow Aut\,(G) \longrightarrow Out(G) \rightarrow 1$$

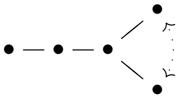
$\Rightarrow$ every involutive automorphism θ of G has a unique image in $Out\,(G)$.

Definition

The image of θ in $Out(G)$ is called the *inner class* of θ.

Suppose is G is simple and simply connected, then the $Out\,(G)$ is isomorphic the automorphism group of the Dynkin diagram of G.

Example: $Out(D_5) = \mathbb{Z}_2$

$Out(D_5) \quad \sim$ 

Atlas understands the inner class as being one

- c : compact inner class
- e : equal rank inner class (same as c)
- s : split inner class
- u : unequal rank inner class (types A_n, D_n, and E_6)
- C : complex inner class (for an identical pair of entries)

Specifying an inner class

It often happens that some of these Atlas inner class specifications coincide.

c $\sim$ e $\sim$ s in types B_n, C_n, E_7, E_8, F_4 and G_2

$\Leftarrow$ $Out\,(G)$ is trivial.

Remark: In fact, except for D_{2n} where $c = s \neq u$, and the complex forms C, the inner classes are **either** s or c.

In the nonsimple cases, one specifies the inner class of G by choosing c, e, s, u for each simple or T_1 factor, or C for each pair of identical simple or T_1 factors.

N.B. when G is not simply connected or adjoint, involutions in the simple factors are not always compatible (i.e., do not factor to G.

In such a case, `atlas` will complain and ask you to make another choice.

After fixing

Step 1: Lie type

Step 2: an isogeny type

Step 3: an inner class of real forms

we end up in `main` mode.

From here we can finally set the real form of the group which want atlas to study. This is done via the `realform` command. Once the `realform` command is entered, `atlas` will present you with a list of the possibilities (actually a list of the real Lie algebras of the possible real forms).

Cartan type	*compact inner class*	*split inner class*	*unequal rank inner class*
A4	su(5), su(4, 1), su(3, 2)	sl(5, R)	(same as split)
B4	so(9), so(8, 1), so(7, 2), so(6, 3), so(4, 5)	(same as compact)	N/A
C4	sp(4), sp(3, 1), sp(2, 2), sp(8, R)	(same as compact)	N/A
D4	so(8), so(6, 2), so*(8)[0, 1], so(8)*[1, 0], so(4, 4)	(same as compact)	so(7, 1), so(5, 3)

Note that there are two isomorphic real forms of D_4.

The reason why there are two copies of $\mathfrak{so}^*(8)$ is that, in fact, there are two distinct, non-conjugate isomorphic subgroups of $SO(8, C)$ (they are related only by an outer automorphism of $SO(8, \mathbb{C})$).

real forms of the exceptional Lie algebras

Cartan type	*compact inner class*	*split inner class*	*unequal rank inner class*
E6	e6, e6(so(10).u(1)), e6(su(6).su(2))	e6(f4), e6(R)	(same as split)
E7	e7, e7(e6.u(1)), e7(so(12).su(2)), e7(R)	(same as compact)	N/A
E8	e8, e8(e7.su(2)), e8(R)	(same as compact)	N/A
F4	f4, f4(so(9)), f4(R)	(same as compact)	N/A
G2	g2, g2(R)	(same as compact)	N/A

Simple Examples

We begin with perhaps the simplest family of examples, tori.

We start by entering the `type` command and telling `atlas` to begin considering a complex torus T

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
```

atlas pauses here as it is waiting for us to specify (if any) a list of the generators of the kernel of the homomorphism

$$\mathbb{C}^\times \to T$$

Generators of finite subgroups of $\mathbb{C}^\times \approx$ elements of $\mathbb{Q}/\mathbb{Z}$

Specify generators of finite subgroup $\Gamma \in \mathbb{C}^\times$ as fractions: e.g.,

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3
3/5
```

atlas then ask for the inner class of real forms with the prompt

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3
3/5

enter inner class(es):
```

Your options here are to enter c or s. If you enter c, you are telling atlas that you wish to consider a compact real torus; i.e., some quotient of the circle group S^1 by a finite subgroup. If you enter s, you are telling atlas that you wish to consider a split real torus; i.e. a quotient of $\mathbb{R}^\times$ by a finite subgroup.

There is another basic type of real tori, those isomorphic to a quotient of $\mathbb{C}^\times$ regarded as a 2-dimensional real Lie group.

Such a torus is setup in atlas by instructing it to consider a quotient of the diagonal subgroup of $\mathbb{C}^\times \times \mathbb{C}^\times$. This done as follows

```
main: type
Lie type: T1.T1
elements of finite order in the center of the simply connected group:
Q/Z.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3,1/3
3/5,3/5

enter inner class(es): C
```

We entered the kernel generators in pairs, with identical rational numbers separated by commas.

This is necessary to ensure that the corresponding finite subgroup of $\mathbb{C}^\times \times \mathbb{C}^\times$ is a subgroup of the diagonal subgroup $\mathbb{C}^\times$.

One constructs more general real tori as products of these three basic types. Thus, for example, a 4-dimensional torus isomorphic to $S^1 \times \mathbb{C}^\times \times \mathbb{R}^\times$ could be setup as follows

```
main: type
Lie type: T1.T1.T1.T1
elements of finite order in the center of the simply connected group:
Q/Z.Q/Z.Q/Z.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): cCs
main: showrealforms
(weak) real forms are:
0: u(1).gl(1,C).gl(1,R)
```

The steps for setting up `atlas` for studying simple Lie groups is very similar.

Example: setting up `atlas` to study $SL(2,\mathbb{R})$.

```
empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
real:
```

Explanation

- the Lie type is given as `A1` since $SL(2,\mathbb{C})$ is of Cartan type A and rank 1.
- we have specified the kernel generators as `sc`. This is telling `atlas` that the real group we are interested in is a real form of the *simply connected* (`sc`) complex group of Lie type `A1`.
  N.B., the real group we end up with, $SL(2,\mathbb{R})$ is not simply connected.
- while have have told `atlas` to choose the split inner class (`s`), we could achieved the same result by specifying the compact inner class (`c`).
  This because $SL(2,\mathbb{C})$ only has one inner class and so the compact and split inner classes coincide.

Example: $SO(5,2)$

```
main: type
Lie type: B3
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2
enter inner class(es): s
main: realform
(weak) real forms are:
0: so(7)
1: so(6,1)
2: so(5,2)
3: so(4,3)
enter your choice: 2
real: components
component group is (Z/2)^1
```

Explanation:

- Since the complexified Lie algebra of $SO(5,2)$ is of Cartan type B and rank 3, we instructed specified `B3` as its Lie type.
- The complex group for which $SO(5,2)$ appears as the set of real points has non-trivial isogeny - and so we specified `1/2` as its kernel generators. This corresponds to the isogeny class of the adjoint group of $\mathfrak{so}(7,\mathbb{C})$. We could also have entered `ad` in response to the request for kernel generators. If we had instead wanted a real form of the simply connected complex group with Lie algebra $\mathfrak{so}(7,\mathbb{C})$ (i.e. $Spin(7,\mathbb{C})$), then we would have entered `0/2`, `sc`, or simply simply hit `Enter` in response to the request for kernel generators.
- At the end of this example, we entered the `real` mode command `components` to demonstrate that `atlas` understands that $SO(5,2)$ is not connected.

Example: $GL(2,\mathbb{R})$

```
empty: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): ss
main: realform
(weak) real forms are:
0: su(2).gl(1,R)
1: sl(2,R).gl(1,R)
enter your choice: 1
real:
```

N.B., we delimit the various factors of the complex group with *periods*, use *commas* to delimit the "components" of the kernel generators in each factor, and then use *no delimiter* at all when specifying the inner class to be taken on each factor.