Lecture 2: `atlas` demo

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The formal construction:

- choose connected, simply connected, complex reductive algebraic group

$$\widetilde{G}_{\mathbb{C}} = G_1 \times G_2 \cdots \times G_k \times \left(\mathbb{C}^{\times}\right)^{\ell}$$

- divide out by finite subgroup Γ of $Z(\widetilde{G}_{\mathbb{C}})$

$$G_{\mathbb{C}} = \widetilde{G}_{\mathbb{C}}/\Gamma$$

- fix compact real form corresponding to anti-holomorphic involutive automorphism $\gamma_o : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ corresponding to a compact real form and a commuting holomorphic involutive automorphism $\theta : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ and set

$$G_{\mathbb{R}} = (G_{\mathbb{C}})^{\gamma_o \theta}$$

Caveat: non-linear covers like $Mp(2n, \mathbb{R})$ can not be constructed this way

Atlas recipe

- specify Lie type of $\widetilde{G}_{\mathbb{C}}$: e.g., A3.T1.D4 for $SL_4 \times \mathbb{C}^{\times} \times SO(8)$
- specify generators of Γ for each simple factor : e.g., 1/4,5/7,1/2,1/2
- specify inner class for each factor : e.g., scc
- choose a real form from list generated by atlas

Example

```
empty: type
Lie type: A3.T1.D4
elements of finite order in the center of the simply connected group:
Z/4.Q/Z.Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/4,5/7,1/2,1/2

enter inner class(es): scs
main: realform
(weak) real forms are:
0: sl(2,H).u(1).so(8)
1: sl(4,R).u(1).so(8)
2: sl(2,H).u(1).so(6,2)
3: sl(2,H).u(1).so*(8)[0,1]
4: sl(2,H).u(1).so*(8)[1,0]
5: sl(4,R).u(1).so(6,2)
6: sl(2,H).u(1).so(4,4)
7: sl(4,R).u(1).so*(8)[0,1]
8: sl(4,R).u(1).so*(8)[1,0]
9: sl(4,R).u(1).so(4,4)
enter your choice:
```

The plan for this lecture is to demonstrate how to utilize the `atlas` to study properties of real groups, so we'll be working mostly in `real` mode today.

More examples can be found on the Atlas Examples webpage

`http://atlas.math.umd.edu/software/examples/`

I also highly recommend checking out Jeff Adams' paper: *Guide to the Atlas Software: Computational Representation Theory of Real Reductive Groups*. This paper can be downloaded from

`http://www.liegroups.org/papers/snowbirdExamples.pdf`

Simplest Examples

We begin with perhaps the simplest family of examples, tori.

We start by entering the `type` command and telling `atlas` to begin considering a complex torus T

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
```

atlas pauses here as it is waiting for us to specify (if any) a list of the generators of the kernel of the homomorphism

$$\mathbb{C}^\times \to T$$

Generators of finite subgroups of $\mathbb{C}^\times \approx$ elements of $\mathbb{Q}/\mathbb{Z}$

Specify generators of finite subgroup $\Gamma \in \mathbb{C}^\times$ as fractions: e.g.,

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3
3/5
```

atlas then ask for the inner class of real forms with the prompt

```
empty: type
Lie type: T1
elements of finite order in the center of the simply connected group:
Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3
3/5

enter inner class(es):
```

Your options here are to enter c or s.

If you enter c, you are telling atlas that you wish to consider a compact real torus; i.e., some quotient of the circle group S^1 by a finite subgroup.

If you enter s, you are telling atlas that you wish to consider a split real torus; i.e. a quotient of $\mathbb{R}^\times$ by a finite subgroup.

There is another basic type of real tori, those isomorphic to a quotient of $\mathbb{C}^\times$ regarded as a 2-dimensional real Lie group.

Such a torus is setup in `atlas` by instructing it to consider a quotient of the diagonal subgroup of $\mathbb{C}^\times \times \mathbb{C}^\times$. This done as follows

```
main: type
Lie type: T1.T1
elements of finite order in the center of the simply connected group:
Q/Z.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/3,1/3
3/5,3/5

enter inner class(es): C
```

We entered the kernel generators in pairs, with identical rational numbers separated by commas.

This is necessary to ensure that the corresponding finite subgroup of $\mathbb{C}^\times \times \mathbb{C}^\times$ is a subgroup of the diagonal subgroup $\mathbb{C}^\times$.

One constructs more general real tori as products of these three basic types. Thus, for example, a 4-dimensional torus isomorphic to $S^1 \times \mathbb{C}^\times \times \mathbb{R}^\times$ could be setup as follows

```
main: type
Lie type: T1.T1.T1.T1
elements of finite order in the center of the simply connected group:
Q/Z.Q/Z.Q/Z.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): cCs
main: showrealforms
(weak) real forms are:
0: u(1).gl(1,C).gl(1,R)
```

Variations on A_1

The following examples will all start with a simple Lie algebra of Cartan type A_1.

$SU(2) / SL(2, \mathbb{R})$

```
empty: type
Lie type: A1 sc s
main: posroots
Name an output file (return for stdout, ? to abandon):
[2]
main: poscoroots
Name an output file (return for stdout, ? to abandon):
[1]
main: components
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
group is connected
real:
```

$SO(3)/SO(2,1)$

```
empty: type
Lie type: A1 ad s
main: posroots
Name an output file (return for stdout, ? to abandon):
[1]
main: poscoroots
Name an output file (return for stdout, ? to abandon):
[2]
main: components
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
component group is (Z/2)^1
real:
```

$SL(2,\mathbb{C})/SO(3,\mathbb{C})$

```
empty: type
Lie type: A1.A1 sc C
main: posroots
Name an output file (return for stdout, ? to abandon):
[0,2]
[2,0]
main: poscoroots
Name an output file (return for stdout, ? to abandon):
[0,1]
[1,0]
```

```
main: type
Lie type: A1.A1 ad C
main: posroots
Name an output file (return for stdout, ? to abandon):
[0,1]
[1,0]
main: poscoroots
Name an output file (return for stdout, ? to abandon):
[0,2]
[2,0]
main:
```

The first thing to point out is that the kernel of the homomorphism

$$SL(2,\mathbb{C}) \times \mathbb{C}^{x} \longrightarrow GL(2,\mathbb{C}) : (X,t) \longrightarrow tX$$

Noting that

$$tX = (-t)(-X)$$

we see that the kernel generator corresponds to the element.

$$\left(\frac{1}{2},\frac{1}{2}\right) \in \mathbb{Z}/2 \times \mathbb{Q}/\mathbb{Z}$$

```
main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): ss
main: showrealforms
(weak) real forms are:
0: su(2).gl(1,R)
1: sl(2,R).gl(1,R)
```

$U(1,1)$

```
main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): cc
main: showrealforms
(weak) real forms are:
0: su(2).u(1)
1: sl(2,R).u(1)
```

Example: $SO(5,2)$

```
main: type
Lie type: B3
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2
enter inner class(es): s
main: realform
(weak) real forms are:
0: so(7)
1: so(6,1)
2: so(5,2)
3: so(4,3)
enter your choice: 2
real: components
component group is (Z/2)^1
```

Explanation:

- Since the complexified Lie algebra of $SO(5,2)$ is of Cartan type B and rank 3, we instructed specified `B3` as its Lie type.
- The complex group for which $SO(5,2)$ appears as the set of real points has non-trivial isogeny - and so we specified `1/2` as its kernel generators. This corresponds to the isogeny class of the adjoint group of $\mathfrak{so}(7,\mathbb{C})$. We could also have entered `ad` in response to the request for kernel generators. If we had instead wanted a real form of the simply connected complex group with Lie algebra $\mathfrak{so}(7,\mathbb{C})$ (i.e. $Spin(7,\mathbb{C})$), then we would have entered `0/2`, `sc`, or simply simply hit `Enter` in response to the request for kernel generators.
- At the end of this example, we entered the `real` mode command `components` to demonstrate that `atlas` understands that $SO(5,2)$ is not connected.

D_{2n} examples

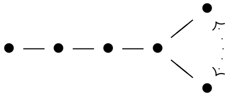
```
empty: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): c
main: showrealforms
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
main:
```

Explanation

Note in the preceding example the occurence of two copies of $\texttt{so}^*(12)$.

Here is what is going on:

`atlas` determines real forms up to conjugacy by G rather than $Aut\,(G)$.

$Out(D_6) \quad \sim$ 

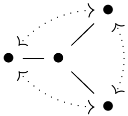
Because of this it counts the two (isomorphic) real forms of $SO^*\,(12)$ separately because they are related (within $SO\,(12, \mathbb{C})$) only by an outer automorphism of $SO\,(2n, \mathbb{C})$.

D_4

The case of $SO(8)$ is even more interesting.

The group of outer automorphisms is S_3 (triality).

$Out(D_4) \quad \sim$



In this case, `atlas` finds

```
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: showrealforms
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
```

Explanation

It turns out that the real forms $Spin\,(6,2)$, $Spin^*\,(8)\,[0,1]$ and $Spin^*\,(8)\,[1,0]$ are all isomorphic groups, corresponding to different, distinct embeddings into $Spin\,(8,\mathbb{C})$.

Remark: By choosing "non-diagonal" quotients of $Spin\,(2n,\mathbb{C})$ by its center, it is possible to construct non-isomorphic real forms.

```
empty: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): c
main: components
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 2
component group is (Z/2)^1
```

```
real: q
main: components
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 3
group is connected
```

Cartan subgroups

Once a real reductive group G has been setup, the `cartan` command provides detailed information about its Cartan subgroups.

Let's begin with a familiar example $SL(2,\mathbb{R})$. But before listing the `atlas` rendition of the Cartan subgroups of $SL(2,\mathbb{R})$, let me just point out that $SL(2,\mathbb{R})$ has two conjugacy classes of Cartan subgroups: the compact Cartan subgroups which are conjugate to

$$T = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

and the split Cartan subgroups which are conjugate to

$$A = \left\{ \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Here is what `atlas` tells us about $SL(2, \mathbb{R})$

```
empty: type
Lie type: A1 sc s
main: cartan
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
```

```
Cartan #0:
split: 0; compact: 1; complex: 0
canonical twisted involution:
twisted involution orbit size: 1;  fiber rank: 1;  #X_r: 2
imaginary root system: A1
real root system is empty
complex factor is empty
real form #1: [0] (1)
real form #0: [1] (1)


Cartan #1:
split: 1; compact: 0; complex: 0
canonical twisted involution: 1
twisted involution orbit size: 1;  fiber rank: 0;  #X_r: 1
imaginary root system is empty
real root system: A1
complex factor is empty
real form #1: [0] (1)
```

Explanation

Atlas identifies two conjugacy classes of Cartans (labeled 0 and 1).

It tells us a lot of other stuff as well. Today I will just point out three simple attributes identified by the software.

A real torus is always isomorphic to a product of the form

$$\left(\mathbb{R}^{\times}\right)^{a} \times \left(S^{1}\right)^{b} \times \left(\mathbb{C}^{\times}\right)^{c}$$

for some integers $a, b, c \in \mathbb{N}$. Here $\mathbb{R}^{\times}$ is the multiplicative group of real numbers, S^1 is the (compact) circle group, and $\mathbb{C}^{\times}$ is the multiplicative group of complex numbers regarded as a real group. The integers which follow `split:`, `compact:`, and `complex:` tell us exactly the isomorphism class of the corresponding Cartan subgroup.

A root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is called *real*, respectively, *imaginary*, if $\alpha(h)$ takes on only real, respectively, pure imaginary, values as h varies over $Lie(H)$. The sets of real and imaginary roots actually form a root subsystem of $\Delta(\mathfrak{g}, \mathfrak{h})$. As shown above, the Cartan types of these subsystems is identified by the `atlas` software. (The complex factor is more complicated to describe - but I should perhaps point out that it does not refer to the roots of $\Delta(\mathfrak{g}, \mathfrak{h})$ that take on complex values.)

Different real forms of a complex group may share a common conjugacy of Cartan subgroups. Indeed, in the equal rank case, every real form contains the (unique conjugacy class of) compact Cartan subgroup.

The `cartan` command only lists the Cartan subgroups for the given real form; but for each of these Cartan subgroups, it lists the other real forms (by number) in which it appears.

```
Cartan #0:
split: 0; compact: 1; complex: 0
canonical twisted involution:
twisted involution orbit size: 1;  fiber rank: 1;  #X_r: 2
imaginary root system: A1
real root system is empty
complex factor is empty
real form #1: [0] (1)
real form #0: [1] (1)

Cartan #1:
split: 1; compact: 0; complex: 0
canonical twisted involution: 1
twisted involution orbit size: 1;  fiber rank: 0;  #X_r: 1
imaginary root system is empty
real root system: A1
complex factor is empty
real form #1: [0] (1)
```

Now for hints as to how to use `atlas` effectively.

Automating input: using Unix pipe commands

- create a text file containing sequence of atlas command to be performed
- use Unix pipe command to feed this to `atlas`

Automation example

To automate

```
binegar@h-c:~$ atlas
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Dec  1 2007 at 09:04:58.
Enter "help" if you need assistance.

empty: type
Lie type: G2 sc s
main: wcells
(weak) real forms are:
0: g2
1: g2(R)
enter your choice: 1
possible (weak) dual real forms are:
0: g2
1: g2(R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon): G2test.out
block: qq
binegar@h-c:~$
```

Automation example, cont'd

1. create file : `G2test.in`

```
type
G2 sc s
wcells
1
1
G2cells.out
qq
```

2. enter command

```
atlas < G2test.in
```

Post-processing `atlas` output

- output to file.
- employ scripting language (e.g. Perl, to reformat output
- employ mathematical software (e.g. Maple, Mathematica, GAP, sage) to analyze data.
- John Stembridge's Coxeter/Weyl package for Maple has been particularly useful to me (caveat: C/W uses non-Bourbaki conventions).

Or, build your own functions inside `atlas` source code.