Lecture 5: Admissible Representations in the Atlas Framework

B. Binegar

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, USA

Nankai Summer School in Representation Theory and Harmonic Analysis
June, 2008

Admissible Reps

Let's begin with

Definition

A (Hilbert space) representation $(\pi, \mathcal{H})$ of a reductive group G with maximal compact subgroup K is called **admissible** if $\pi|_K$ acts by unitary operators on $\mathcal{H}$ and if each K-type (i.e each distinct equivalence class of irreducible representations of K) occurs with finite multiplicity.

basic idea:

The representations we are most interested in, the unitary irreducible ones, behave very nicely upon restriction to K (in particular, they are all admissible).

The admissible representations also have these nice properties and are a "smoother" class of representations to work in: they have a uniform parameterization and construction

The Langlands Classification

Theorem (Langlands, 1973)

Fix a minimal parabolic subgroup $P_o = M_o A_o N_o$. *Then equivalence classes of irreducible admissible representations of* G *are in a one-to-one correspondence with the set of triples* $(P, [\sigma], \nu)$ *such that*

- P *is a parabolic subgroup of* G *containing* P_o
- σ *is an equivalence class of ireducible tempered unitary representations of* M *and* $[\sigma]$ *is its equivalence class*
- ν *is an element of* $\mathfrak{a}'$ *with* $Re(\nu)$ *in the open positive chamber.*

The correspondence is

$$(P, [\sigma], \nu) \longleftrightarrow \text{unique irreducible quotient of } Ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$$

Other "Langlands" parameterizations

- tempered representations $\longrightarrow$ discrete series representations (Knapp-Zuckerman)
- L-groups and L-data (Adams-Vogan)
- Adams-du Cloux - two-sided parameter space $\leftrightarrow$ pairs (x, y) of KGB orbits $(x \in K\backslash G/B \ , \ y \in K^\vee\backslash G^\vee/B^\vee)$
- `atlas` - computational enumeration of representations

Remark: While conceptually, a simple enumeration of representations is about as poor a parameterization as one could imagine - `atlas` also provides a lot of data about the properties of the representations it finds.

In fact: `atlas`'s simple enumeration of the irreducible admissible representations leads to a certain conceptual advantage:

representations are stripped of their constructive pedigrees, and certain algebraic invariants and relationships placed are placed in high relief.

The main thrust of the remaining set of lectures will be to demonstrate how `atlas` enables one to uncover a certain, natural, algebraic taxonomy of the set $\widehat{G}_{adm}$ by analyzing the `atlas` output.

The algebraic setting

Our first task is to explain how one transfers the study of $\widehat{G}_{adm}$ to an algebraic setting.

Definition

Let $(\pi, \mathcal{H})$ be an representation of a reductive group G on a Hilbert space $\mathcal{H}$ and let K be a maximal compact subgroup of G. A vector $v \in \mathcal{H}$ is said to be *K-finite* if $\{\pi(k)v \mid k \in K\}$ spans a finite-dimensional subspace of $\mathcal{H}$.

In restricting attention to K-finite vectors we not only regularize the action of a maximal compact subgroup K:

Theorem

If $(\pi, \mathcal{H})$ is an admissible representation of G, then every K-finite vector is a smooth vector and, moreover, the space $\mathcal{H}_{K-finite}$ of K-finite vectors in $\mathcal{H}$ is stable under $\pi(\mathfrak{g})$.

Upshots:

- The set of K-finite vectors of an admissible representation carries both a representation of the Lie algebra $\mathfrak{g}$ of G and a unitary (but highly reducible) representation of K.
- The K action retains a lot of information about the original G action.
- It fact almost all of the distinguishing properties of an admissible representation of a reductive group G are encoded in its set of K-finite vectors.

Before making use of the K-finite vectors, we formalize the algebraic setting as follows:

Definition

A $(\mathfrak{g}, K)$-**module** is a complex vector space V carrying both a Lie algebra representation of $\mathfrak{g}$ and a group representation of K such that

- The representation of K on V is locally finite and smooth.
- The differential of the group representation of K coincides with the restriction of the Lie algebra representation to $\mathfrak{k}$.
- The group representation and the Lie algebra representations are compatible in the sense that
$$\pi_K(k)\,\pi_{\mathfrak{k}}(X) = \pi_{\mathfrak{k}}(Ad(k)\,X)\,\pi_K(k) \quad .$$

Definition (Theorem)

Suppose (π, V) is a smooth representation of a reductive Lie group G, and K is a compact subgroup of G. Then the space of K-finite vectors can be endowed with the structure of a $(\mathfrak{g}, K)$-module. We call this $(\mathfrak{g}, K)$-module the **Harish-Chandra module** of (π, V).

Remark: Henceforth, we shall be rather indiscriminant in our language; using irreducible admissible representations, irreducible $(\mathfrak{g}, K)$-modules and irreducible Harish-Chandra modules as synonymous terms with highlighting slightly different points of view and utility.

Reduction to infinitesimal character ρ

We will now begin organizing the set of $\widehat{G}_{adm}$ of irreducible admissible representations into smaller subsets characterized by certain invariants of their underlying Harish-Chandra modules.

Let V be the H-C module of an irreducible representation $\pi \in \widehat{G}_{adm}$

in particular, $V \to$ irr rep of $U(\mathfrak{g})$

$\Rightarrow \forall\ z \in Z(\mathfrak{g})$, $\pi(z)$ acts by some scalar on V. (Schur's lemma)

The map

$$Z(\mathfrak{g}) \to \mathbb{C} : z \mapsto \pi(z) = \phi(z) Id$$

thus induces a homomorphism

$$\phi : Z(\mathfrak{g}) \to \mathbb{C}$$

Fix $\mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$. There is a natural homomorphism (the Harish-Chandra homomorphism)

$$\gamma_{HC} = Z(\mathfrak{g}) \to S(\mathfrak{h})$$

and, in fact, every homomorphism

$$Z(\mathfrak{g}) \to \mathbb{C}$$

corresponds to the evaluation of a Harish-Chandra homomorphism γ_{HC} at a point $\lambda \in \mathfrak{h}^*$.

Definition

The point $\lambda \in \mathfrak{h}^*$ and is called the **infinitesimal character** of π (and/or V). It is unique up to an action of the Weyl group of $\mathfrak{g}$.

Translation Principle

Let $\widehat{G}_{adm,\lambda}$ denote the set of irreducible admissible representations of G with infinitesimal character λ

Each $\widehat{G}_{adm,\lambda}$ is a finite set and moreover

$$\widehat{G}_{adm} = \coprod_{\lambda \in \mathfrak{h}^*/W} \widehat{G}_{adm,\lambda}$$

The Janzten-Zuckerman Translation Principle: The internal structure of $\widehat{G}_{adm,\lambda}$ can be "translated" to that of $\widehat{G}_{adm,\mu}$ whenever $\lambda - \mu$ is an element of the weight lattice of G.

Upshot: suffices to understand the structure of $\widehat{G}_{adm,\lambda}$ on representatives of finitely many "translation families" of infinitesimal character.

In fact: suffices to understand the structure of $\widehat{G}_{adm,\rho}$.

$$\rho = \text{half sum of positive roots} \leftrightarrow \text{inf char of trivial rep}$$

It is in this setting that the `atlas` software actually carries out its computations. (Jeff Adams has written Perl scripts for translating results of `atlas` computations to arbitrary regular integral infinitesimal character.)

Blocks of irreducible Harish-Chandra modules

Lurking "under the hood" of the `atlas` software is a parameterization of $\widehat{(G)}_{adm,\rho}$ in terms of a pair $(x, y) \in K\backslash G/B \times K^\vee\backslash G^\vee/B^\vee$

(There is a certain compatibility condition between x and y.)

$K \leftrightarrow$ the choice of real reductive group

$K^\vee \leftrightarrow$ a choice of a real form of the complex dual group $G^\vee$

Definition

A *block* of representations is set of representations for which the KGB parameters (x, y) range over $K\backslash G/B \times K^\vee\backslash G^\vee/B^\vee$ corresponding to fixed real forms of G and $G^\vee$.

Example: the blocks of E8

Below is a table is listing the number of elements in each "block"' ' of E_8:

	e_8	$E_8\left(e_7, su\left(2\right)\right)$	$E_8\left(\mathbb{R}\right)$
e_8	0	0	1
$E_8\left(e_7, su\left(2\right)\right)$	0	3150	73410
$E_8\left(\mathbb{R}\right)$	1	73410	453060

The total number of equivalences classses irreducible Harish-Chandra modules of the split form $E_8\left(\mathbb{R}\right)$ with infinitesimal character ρ is thus

$$1 + 73410 + 453060 = 526\,471$$

i.e., the sum of three blocks corresponding to the three "dual real forms" of $E_8\left(\mathbb{C}\right)^\vee \approx E_8\left(\mathbb{C}\right)$.

Let HC_λ be the set of irreducible Harish-Chandra modules of infinitesimal character λ.

Definition

Given two objects X, Y in HC_λ, we say $X \rightsquigarrow Y$ if there exists a finite-dimensional representation F of G appearing in the tensor algebra $T(\mathfrak{g})$ such that Y appears as a subquotient of $X \otimes F$. Write $X \sim Y$ if $X \rightsquigarrow Y$ and $Y \rightsquigarrow X$. The equivalence classes for the relation $\sim$ are called **cells** (of Harish-Chandra modules).

It turns out that the decomposition of HC_λ into disjoint cells is compatible with the decomposition into blocks. Moreover, if one thinks of the relations $X \rightsquigarrow Y$ as defining a directed graph structure on HC_λ, then

- blocks of representations $\leftrightarrow$ the connected components of the graph
- cells of representations $\leftrightarrow$ strongly (i.e. bidirectionally) connected components of the graph.

The `atlas` software explicitly computes this graph structure as a *by-product* of its computation of the *KLV*-polynomials.

In fact, `atlas` computes an even more elaborate graph structure:

Definition

Consider a block B of irreducible Harish-Chandra modules of infinitesimal character ρ. The W-**graph** of B is the weighted graph where:

- the vertices are the elements $v \in B$
- there is an edge (v, v') of **multiplicity** m between two vertices if

$$\text{coefficient of } q^{\left(|v|-|v'|-1\right)/2} \text{ in } P_{v,v'}(q) = m \neq 0$$

- there is assigned to each vertex v a subset $\tau(v)$ of the set of simple roots of $\mathfrak{g}$ (the **descent set** of v).

Example: $G2$

Below is an example of the (annotated) output of `wgraph` for the big block of G_2.

```
block    descent      edge vertices,
element    set        multiplicities
   0       {}              {}
   1      {2}           {(3,1)}
   2      {1}           {(4,1)}
   3      {1}       {(0,1),(1,1),(6,1)}
   4      {2}       {(0,1),(2,1),(5,1)}
   5      {1}         {(4,1),(8,1)}
   6      {2}         {(3,1),(7,1)}
   7      {1}         {(6,1),(11,1)}
   8      {2}         {(5,1),(10,1)}
   9     {1,2}        {(7,1),(8,1)}
  10      {1}           {(8,1)}
  11      {2}           {(7,1)}
```

Example cont'd: the W-graph of $G2$

The W-graph for this block thus looks like

$$\begin{tikzcd}
 & 11,\{2\} \arrow[d, shift left, "1"] & 9,\{1,2\} \arrow[dl, "1"'] \arrow[dr, "1"] & 10,\{1\} \arrow[d, shift left, "1"] & \\
 & 7,\{1\} \arrow[u, shift left, "1"] \arrow[d, shift left, "1"] & & 8,\{2\} \arrow[u, shift left, "1"] \arrow[d, shift left, "1"] & \\
 & 6,\{2\} \arrow[u, shift left, "1"] \arrow[d, shift left, "1"] & & 5,\{1\} \arrow[u, shift left, "1"] \arrow[d, shift left, "1"] & \\
1,\{2\} \arrow[r, shift left, "1"] & 3,\{1\} \arrow[l, shift left, "1"] \arrow[u, shift left, "1"] \arrow[dr, "1"] & & 4,\{2\} \arrow[u, shift left, "1"] \arrow[r, shift left, "1"] \arrow[dl, "1"'] & 2,\{1\} \arrow[l, shift left, "1"] \\
 & & 0,\{\} & &
\end{tikzcd}$$

Note that there are four cells: $\{0\}$, $\{1,6,7,11\}$, $\{2,4,5,8,10\}$, and $\{9\}$.

I'll now briefly introduce the basic algebraic invariants which we shall eventually attach to the representations of a cell.

Definition

Let V be an irreducible Harish-Chandra module. The set

$$Ann(V) = \{X \in U(\mathfrak{g}) \mid Xv = 0 \quad , \quad \forall\ v \in V\}$$

is a two-sided ideal in $U(\mathfrak{g})$. It is called the *primitive ideal* in $U(\mathfrak{g})$ attached to V.

Fact: If two irreducible representations share the same annihilator in $U(\mathfrak{g})$, then they must have the same infinitesimal character.

Set
$Prim(\mathfrak{g}) :=$ set of primitive ideals in $U(\mathfrak{g})$

$Prim(\mathfrak{g})_\lambda :=$ set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character λ

Then

$$Prim\,(\mathfrak{g}) = \coprod_{\lambda \in \mathfrak{h}^*/W} Prim\,(\mathfrak{g})_\lambda$$

The correspondence

$$HC_\lambda \to Prim\,(\mathfrak{g})_\lambda : \pi \mapsto Ann(\pi)$$

is often one-to-one, but generally speaking, several-to-one.

$\Rightarrow$ a fairly fine grained-partitioning of HC_λ

$U(\mathfrak{g})$ is naturally filtered according to

$$U^n(\mathfrak{g}) = \{X \in U(\mathfrak{g}) \mid X = \text{product of} \leq n \text{ elements of } \mathfrak{g}\}$$

Moreover, one has

$$U^n(\mathfrak{g})U^m(\mathfrak{g}) \subseteq U^{n+m}(\mathfrak{g})$$

The graded algebra

$$gr(U(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} \left(U^n(\mathfrak{g}) \,/\, U^{n-1}(\mathfrak{g})\right)$$

is well defined, and, in fact

$$gr\left(U\left(\mathfrak{g}\right)\right) \approx S\left(\left(\mathfrak{g}\right)\right)$$

If J is a primitive ideal in $U(\mathfrak{g})$, then $gr(J)$ is a prime ideal in $S(\mathfrak{g})$,

$\rightarrow$ an ideal of polynomials over $\mathfrak{g}^*$.

$\rightarrow$ an affine variety

$$\mathcal{V}(J) = \{\lambda \in \mathfrak{g}^* \mid \phi(\lambda) = 0 \ \forall \phi \in gr(J)\}$$

Definition

Let J be a primitive ideal. The variety $\mathcal{V}(J)$ is called the *associated variety* of J.

Fundamental Facts:

$\mathcal{V}(J)$ is a closed, G-invariant subset of $\mathfrak{g}^*$.

In fact, $\mathcal{V}(J)$ is the Zariski closure of a single nilpotent orbit in $\mathfrak{g}^*$

Definition

The *nilpotent orbit attacted to J* is the unique dense orbit in $\mathcal{V}(J)$.

Taking J to be the annihilator of an irreducible Harish-Chandra module V, one can thus speak of the *associated variety* of V and the *nilpotent orbit attached* to V.

However, we will refrain from doing so because there are more refined notions of associated variety and attached nilpotent orbit that are capable of distinguishing $(\mathfrak{g}, K)$-modules with the same annihilator.

The Weyl group of $\mathfrak{g}$ acts naturally on the Grothendieck group $\mathbb{Z}HC_\lambda$ of irreducible Harish-Chandra modules of infinitesimal character λ via the "coherent continuation representation".

cells of HC modules $\rightarrow$ semisimple quotient representations $\leftrightarrow$ *cell representations*

Fact: the W-representation carried by a cell is encoded in its W-graph.

Let
$\Pi = \{$Bourbaki indices of simple roots$\}$
$V = \{v[i] \mid 1 \leq i \leq n\}$: an enumeration of the cell vertices
$M : n \times n$ matrix whose (i, j) entry is the multiplicity of the edge $v[i] \rightarrow v[j]j$
τ : map from $I \rightarrow 2^\Pi$ such that $\tau(i) =$ tau invariant of $v[i]$

Cell Representations

The action of a simple reflection corresponding to a simple root i on cell representation corresponds to

$$T_i v_k = \begin{cases} v[k] & \text{if } i \notin \tau(k) \\ -v[k] + \sum_{l : i \notin \tau(l)} M_{kl} v[l] & i \notin \tau(k) \end{cases}$$

The W-representation carried by a cell can be computed by evaluating

$$\chi_C(s_i \cdots s_j) = trace\,(T_i \cdots T_j)$$

on a representative $s_i \cdots s_j$ of each conjugacy class and then decomposing this character into a sum of irreducible characters.

Upshot: The cell representations are computable from the `atlas` output.