Lecture 6: Cells and Orbits

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The Atlas Setting

- $G_{\mathbb{R}}$: a real reductive Lie group realizable as the set of real points of a reductive algebraic group defined over $\mathbb{R}$;
- $\widehat{G_{\mathbb{R}}}_{,adm}$: set of equivalence classes of irr admissible reps
- $\mathcal{L}_\lambda$: a set of Langlands parameters for irr admissible reps of **regular integral infinitesimal character** λ (a finite set).
- $\mathcal{HC}_\lambda = \left\{ V_x = \pi_x|_{K\text{-finite}} \mid x \in \mathcal{L}_\lambda \right\}$: set of irreducible Harish-Chandra modules corresponding to the irr admissible reps $\pi_x \in \widehat{G_{\mathbb{R}}}$, $x \in \mathcal{L}_\lambda$.

The Atlas software catalogs and analyzes reps in $\mathcal{HC}_\lambda$.

Notation / Apparatus

- $\mathfrak{g} = Lie\,(G_{\mathbb{R}})_{\mathbb{C}}$; $\mathfrak{h}$, a CSA for $\mathfrak{g}$;
  $\Delta = \Delta\,(\mathfrak{h}, \mathfrak{g})$, roots of $\mathfrak{h}$ in $\mathfrak{g}$;
  $\Pi \subset \Delta$, choice of simple roots in Δ;
- G : adjoint group of $\mathfrak{g}$
- $\mathcal{N}_{\mathfrak{g}}$: nilpotent cone in $\mathfrak{g}$ (identifying $\mathfrak{g}^*$ with $\mathfrak{g}$)
- Set $\mathcal{S} \equiv \{$special nilpotent orbits$\}$
- $d : G\backslash\mathcal{N}_{\mathfrak{g}} \to G\backslash\mathcal{N}_{\mathfrak{g}}$: the Spaltenstein-Barbasch-Vogan duality map that restricts to an involution on $image(d) \equiv \mathcal{S} \equiv$ set of **special** nilpotent orbits.

Definition: Let $x, y \in \mathcal{HC}_\lambda$. Write $x \rightharpoonup y$ if there exists a f.d. rep F occurring in $T(\mathfrak{g})$ such that

$$y \text{ occurs as subquotient of } x \otimes F$$

A **cell** of H-C modules is a maximal collection of $x \in \mathcal{HC}_\lambda$ such that

$$x, y \in C \quad \Longrightarrow \quad x \rightharpoonup y \text{ and } y \rightharpoonup x$$

Basic facts:

(i) $x \in \mathcal{HC}_\lambda \Rightarrow Ann_{U(\mathfrak{g})}(x)$ primitive ideal of reg int char
$\Rightarrow gr(Ann(x)) \sim$ ideal in $S(\mathfrak{g}$
$\Rightarrow$ associated variety $AV(Ann(x)) \in \mathfrak{g}^*$
Fact: λ reg integral $\Rightarrow AV(Ann(x)) = \overline{\mathcal{O}}$ the closure of a special orbit

(ii) $x, y \in C \Longrightarrow \mathcal{O}_x = \mathcal{O}_y \equiv \mathcal{O}_C$

(iii) $x \in \mathcal{L}_\lambda$, with λ regular, integral inf. char.
$\Longrightarrow \quad \mathcal{O}_x$ is **special** nilpotent orbit.

Problem: which cells of reps correspond to which special nilpotent orbits?

Key Fact:

The W-rep carried by a cell (induced from coherent cont rep on block) has a unique **special** constituent σ_C.

This coincides with special W-rep attached to $\mathcal{O}_C$ via Springer correspondence.

Atlas Output

The Atlas software not only catalogs the KLV polynomials for the representations in $\mathcal{L}_\rho$, it computes the entire W-graph of $\mathcal{L}_\rho$: a weighted directed graph such that

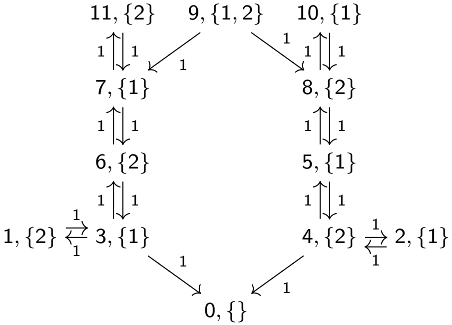
- vertices $\leftrightarrow x \in \mathcal{L}_\rho$
- vertex weights $\leftrightarrow$ descent sets $\tau(x)$ of $x \in \mathcal{L}_\rho$
  For each $x \in \mathcal{L}_\lambda$, $\tau(x)$ is a certain subset of Π
  $\tau(x)$ is the tau invariant of $Ann(V_x)$.
- edges $\leftrightarrow$ relations $y \to x \quad \equiv V_y$ occurs in $V_x \otimes \mathfrak{g}$
- edge multiplicities: $mult(y \to x) =$ multiplicity of V_y in $V_x \otimes \mathfrak{g}$

H-C cells correspond to bidirectionally connected subgraphs

Example: the big block of the split real form of G_2.

block element	descent set	(edge vertex,multiplicity)
0	{}	{}
1	{2}	{(3,1)}
2	{1}	{(4,1)}
3	{1}	{(0,1),(1,1),(6,1)}
4	{2}	{(0,1),(2,1),(5,1)}
5	{1}	{(4,1),(8,1)}
6	{2}	{(3,1),(7,1)}
7	{1}	{(6,1),(11,1)}
8	{2}	{(5,1),(10,1)}
9	{1,2}	{(7,1),(8,1)}
10	{1}	{(8,1)}
11	{2}	{(7,1)}

The W-graph for this block thus looks like



Cell #	Members
0	0
1	1, 3, 6, 7, 11
2	2, 4, 5, 8, 10
3	9

Theorem. (Spaltenstein, Vogan) Suppose C is a cell of H-C modules with associated special nilpotent orbit $\mathcal{O}_C$ and let $\mathfrak{l}$ be a (standard) Levi subalgebra of $\mathfrak{g}$. Then

$$\mathcal{O}_C \subset \overline{ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}}\right)} \quad \Longleftrightarrow \quad \exists\, x \in C \;\; s.t.\, \Pi_{\mathfrak{l}} \subset \tau(x)$$

where $\Pi_{\mathfrak{l}} =$ the simple roots of $\mathfrak{l}$. Here $\Pi_{\mathfrak{l}} \subset \Pi_{\mathfrak{g}}$ and

$$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}}\right) \equiv \text{ unique dense orbit in } G \cdot \mathfrak{n}$$

where $\mathfrak{n}$ is nilradical of any parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$.

Orbits of the form $ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}}\right)$ are called *Richardson orbits*.

Upshot: tau invariants of a cell constrain which Richardson orbit closures can contain $\mathcal{O}_C$

Problem: Every Richardson orbit is special, but not every special orbit is Richardson.

How do we separate configurations like

$$\begin{array}{c} \mathcal{O}_C = \mathcal{O}_{Richardson} \\ \uparrow \\ \mathcal{O}_{C'} = \mathcal{O}_{special} \end{array}$$

S-V criterion would only tell us that both $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ are contained in $\overline{\mathcal{O}_{Richardson}}$

Levi subalgebras and Richardson orbits

- $\Gamma \subset \Pi$: a subset of the simple roots.
- $\mathfrak{l}_\Gamma$: standard Levi subalgebra attached to

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \langle \Gamma \rangle} \mathfrak{g}_\alpha$$

- $R_\Gamma = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} \left(\mathbf{0}_{\mathfrak{l}_\Gamma} \right)$: the Richardson orbit induced from the trivial orbit of a Levi subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$

Fact: every special orbit $\mathcal{O}$ is determined by

(i) the Richardson orbits that contain $\mathcal{O}$

(ii) the Richardon orbits that contain $d(\mathcal{O})$

David Vogan's Idea: The tau invariants of a cell should tell us which Richardson orbits contain $\mathcal{O}_C$ and which Richardson orbits contain the SBV dual of $\mathcal{O}_C$.

Tau signatures for cells

Set

$$\tau(C) \equiv \{\tau(x) \mid x \in C\}$$

Facts

- $\#$ distinct $\tau(C) = \#$ special nilpotent orbits
- Let

$$\tau^{\vee}(C) = \{\Pi - \tau(x) \mid x \in C\}$$

$\Longrightarrow$ duality operation for tau sets.

Definition:

$\Psi = \{\Gamma \subset \Pi\}$: a set of *standard* Γ*'s*: a collection of $\Gamma \in 2^{\Pi}$ such that

$$i : \Psi \leftrightarrow \{\text{conjugacy classes of Levi subalebras}\}$$

is a bijection. (E.g., choose std Γ's to be first in the lexicographic ordering of their W-conj class)

Let $\Gamma, \Gamma' \in \Psi$ and let $\mathfrak{l}_\Gamma$ and $\mathfrak{l}_{\Gamma'}$ be the corresponding standard Levi subalgebras of $\mathfrak{g}$. We shall say

$$\Gamma \leq \Gamma' \Longleftrightarrow ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}) \subset \overline{ind_{\mathfrak{l}_{\Gamma'}}^{\mathfrak{g}}(\mathbf{0})}$$

Remark: this ordering tends to reverse the ordering by cardinality.

Definition: The **tau signature** of an H-C cell C is the pair

$$\tau_{sig}(C) \equiv \left(min\left(\tau(C) \cap \Psi\right)\ ,\ min\left(\tau^{\vee}(C) \cap \Psi\right)\right)$$

Definition: Let $\mathcal{O}$ be a special orbit. The *tau signature* of $\mathcal{O}$ is the pair $(\tau(\mathcal{O}), \tau^\vee(\mathcal{O}))$ where

$$\tau(\mathcal{O}) = min\left\{\Gamma \in \Psi \mid \mathcal{O} \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})}\right\}$$

$$\tau^\vee(\mathcal{O}) = min\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})}\right\}$$

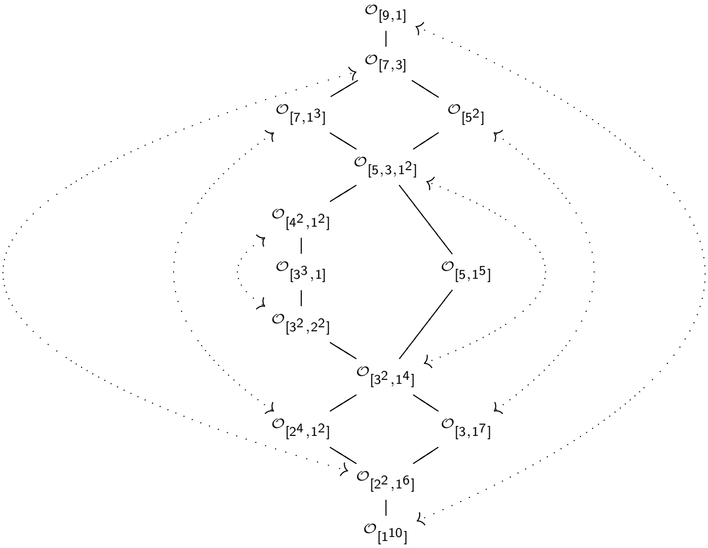
The point: we are using pairs of subsets of simple roots to tell us when a Richardson orbit closure can contain a special orbit (or its dual).

The same kind of pairs tells us when the orbit attached to a cell can be contained in Richardson orbit (or when the dual cell can be contained in the closure of Richardson orbit).

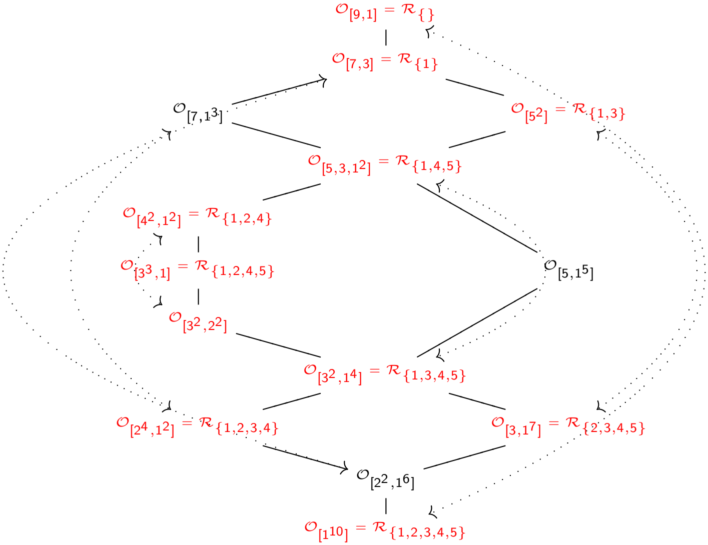
Corollary (to S-V criterion)

$$\mathcal{O}_C = \mathcal{O} \iff \tau_{sig}(C) = \tau_{sig}(\mathcal{O})$$

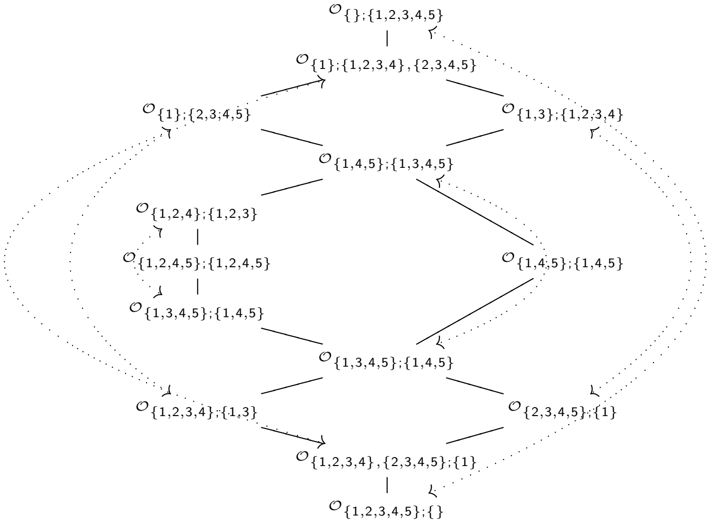
Example: Special Orbits of D_5



Richardson Orbits of D_5



Tau Signatures of Special Orbits of D_5



Tau signatures for cells in the big block of $SO(5,5)$

- 365 representations with inf. char. ρ in big block
- 32 cells in the big block

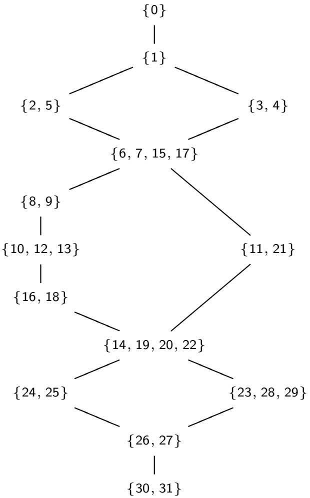
Output of `extract-cells`

```
// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {2} --> 1,2
1[3]: {1} --> 0
2[5]: {3} --> 0,3,4
3[13]: {5} --> 2
4[14]: {4} --> 2
  *
  *
  *
// cell #29:
0[328]: {1,2,4,5} --> 2,3
1[340]: {2,3,4,5} --> 2
2[358]: {1,3,4,5} --> 0,1
3[364]: {1,2,3} --> 0

// cell #30:
0[353]: {1,2,3,4,5}

// cell #31:
0[357]: {1,2,3,4,5}
```

cell #	tau signature
0	{} , {1,2,3,4,5}
1	{1} , {1,2,3,4}
2	{1} , {2,3,4,5}
3	{1,3} , {1,3,4,5}
*	*
*	*
*	*
28	{2,3,4,5} , {1}
29	{2,3,4,5} , {1}
30	{1,2,3,4,5} , {}
31	{1,2,3,4,5} , {}

Each of these coincides with the tau signature of a particular nilpotent orbit.

Cell-Orbit Correspondences for $SO(5,5)$



More Generally:

Exceptional Groups: tables by Spaltenstein list induced orbits, and Hasse diagrams.

Tau signatures of special orbits can be done by hand.

1. Use Spaltenstein's tables to figure out which special orbits are Richardson orbits and to identify the std Γ's corresponding to the corresponding Levi subalgebra.

2. Place the Richardson orbits in the Hasse diagram of special orbits, and then figure out the Γ parameters of the minimal Richardson orbits that contain a given special orbit and the minimal Richardson orbits that contain its Spaltenstein dual

Even E_8 can be done by hand.

Classical Groups:

Partition classification $\longrightarrow$ closure relations

Just need to

- which partitions correspond to special orbits (easy recipes in Collingwood-McGovern)
- use dominance ordering of partitions to partial order special orbits
- use formulas in [C-M] to determine partitions corresponding to Richardson orbits for each $\Gamma \in \Psi$. Place these in the Hasse diagram of special orbits and at the same time partial order Ψ.
- Use the partial ordering of Ψ to ascribe tau signatures to cells (employing `atlas` data)
- match orbit tau sigs to cell tau sigs

Cell-Orbit correspondences have now been computed for all exceptional and classical

Conclusion:

- Atlas data $\Longrightarrow$ algorithm mapping Langlands parameters to nilpotent orbits. Key is to first collect Langlands parameters into cells.
- Can one actually identify even finer invariants?
  - Can one tell when $Ann(V_x) = Ann(V_y)$? (yes!).
  - What about the associated variety of V_x (union of $K_{\mathbb{C}}$-orbits)?
- Are there representation theoretical intepretations of other combinatorial aspects of W-graphs?

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Acknowledgements

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