Lecture 7: HC Cells and Primitive Ideals

B. Binegar

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, USA

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Set

$\mathfrak{g}$: complex semisimple Lie algebra

$U(\mathfrak{g})$: the universal enveloping algebra of $\mathfrak{g}$.

V : a left $U(\mathfrak{g})$-module.

$Ann(V)$: the **annihilator** of V, the two-sided ideal $Ann(V)$ defined by

$$Ann(V) = \{x \in U(\mathfrak{g}) \mid xv = 0 \ , \ \forall \ v \in V\}$$

Definition

A **primitive ideal** is the annihilator of an irreducible $U(\mathfrak{g})$-module.

$Prim(\mathfrak{g}) :=$ the set of primitive ideals of $U(\mathfrak{g})$.

If V is an irreducible $U(\mathfrak{g})$-module, the center of $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character χ_λ; and in fact,

$$Ann(V) \cap Z(\mathfrak{g}) = ker\chi_\lambda$$

Collecting together the set of primitive ideals having the same infinitesimal character we have

$$Prim(\mathfrak{g}) = \coprod_{\lambda \in \mathfrak{h}^*/W} Prim(\mathfrak{g})_\lambda$$

where

$$Prim(\mathfrak{g})_\lambda = \{\text{primitive ideals with central character } \lambda\}$$

Highest weight modules

Irr HC-modules $\to$ a particular family of irr $U(\mathfrak{g})$-modules that arises naturally in the study of $\widehat{G}_{\mathbb{R},adm}$

irr HW modules, however, turns out to be a much more convenient family for discussing primitive ideals.

Set

$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$: Borel subalgebra of $\mathfrak{g}$

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \alpha$

Definition

A $\mathfrak{g}$-module V is said to be a **highest weight module** (w.r.t. $\mathfrak{b}$) if $\exists v \in V \; s.t. V = U(\mathfrak{g})v \; and \; Xv = 0 \; \forall \; X \in \mathfrak{n}$

Irr HW modules have a uniform algebraic construction in terms of Verma modules. (analogous to construction of irreducible adm reps in terms of *standard* reps)

Verma modules

Definition

Let $\lambda \in \mathfrak{h}^*$. The **Verma module** $M(\lambda)$ of highest weight $\lambda + \rho$ is the left $U(\mathfrak{g})$-module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda+\rho}$$

Here $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$, and $\mathbb{C}_{\lambda+\rho}$ is the 1-dimensional representation of $\mathfrak{b}$ defined by

$$(h + x) v = (\lambda + \rho)(h) v \quad \forall\ h \in \mathfrak{h}\ ,\ x \in \mathfrak{n}\ ,\ v \in \mathbb{C}_{\lambda+\rho}$$

Theorem

Let $\lambda \in \mathfrak{h}^*$.
(i) The Verma module $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$ which is of highest weight $\lambda + \rho$.
(ii) Every irreducible highest weight module is isomorphic to some $L(\lambda)$.

Duflo's theorem

Theorem (Duflo)

For $w \in W(\mathfrak{g}, \mathfrak{h})$ *set*

$$M_w = M(w\rho - \rho)$$

and let

$$L_w = \textit{unique irreducible quotient of } M_w$$

Then

$$\varphi : W \longrightarrow Prim(\mathfrak{g})_\rho : w \longrightarrow Ann(L_w)$$

is a surjection.

Parameterizing $Prim(\mathfrak{g})_\rho$ is tantamount to understanding the fiber of $\phi : W \longrightarrow Prim(\mathfrak{g})_\rho$

Left Cells in W

Definition

Let $\sim$ be the equivalence relation on W defined by

$$w \sim w' \quad \Longleftrightarrow \quad \varphi(w) = \varphi(w')$$

The corresponding equivalence classes of elements of W are called **left cells** in W.

In order to make contact with the special representations of the Weyl group, we need a technical device due to Joseph.

$U(\mathfrak{g})/J$ prime Noetherian ring

$\Rightarrow U(\mathfrak{g})/J \approx Mat(n \times n, D)$ for some skew field D. $n \equiv GoldieRank(J)$.

Theorem

For $\mu \in \mathfrak{h}^*$ set

$$p(\mu) = GoldieRank(U(\mathfrak{g})/Ann(L(\mu)))$$

Then for $w \in W$, $p(w\mu)$ is a harmonic polynomial on $P(\Delta)^{++}$ (the dominant regular chamber). Moreover, if w, w' belong to the same left cell $\mathcal{C}$

$$p_{w'} = p_w$$

each Goldie rank polynomial p_w is a harmonic polynomial on $\mathfrak{h}^*$, $\Rightarrow W$ acts p_w and thereby generates an irr rep W.

Definition

Fix a finite-dimensional representation σ of W. The $w \in W$ such

$$\mathbb{C}\langle W \cdot p_w \rangle \approx \sigma$$

comprise the **double cell** in W corresponding to $\sigma \in \widehat{W}$. The representations of W that arise in this fashion are called **special representations** of W.

Theorem

If w, w' belong to same double cell corresponding to a special representation $\sigma \in \widehat{W}$. $Ann\,(L_w)$ and $Ann((L_{w'})$ share the same associated variety.
The unique dense orbit in $AV(Ann(L_w))$ is a special nilpotent orbit $\mathcal{O}$ and the W-rep attached $\mathcal{O}$ by the Springer correspondence coincides with σ.

Theorem

Let $C \subset W$ be a double cell and let $\sigma \in \widehat{W}$ be the assoc (special) W rep. Then

$$Card\,\{Ann(L_w \mid w \in C\}\dim\sigma$$

Two pictures at inf char ρ

Primitive ideals and HW modules

W	$\{Ann(L_w) \mid w \in W\}$	same inf char
$\cup$	$\cup$	
C : *dbl cell*	$\{Ann(L_w) \mid w \in C\}$	same nilpotent orbit
$\cup$	$\cup$	
c : *left cell*	$\{Ann(L_w) \mid w \in c\}$	same primitive ideal

HC modules

block of HC modules	same inf char
$\cup$	
cell of HC modules	same nilpotent orbit
$\cup$	
???	same primitive ideal

Let M_w be the Verma module of highest weight $w\rho - \rho$, containing $L(w\rho)$ as its irreducible quotient.

Let $I_w = Ann(L(w\rho))$

M_e corresponds to the Verma module with the trivial representation as its irreducible quotient. $Ann(L_e)$ is *augmentation ideal*, the maximal ideal in $Prim_\rho$

$$Ann(L_w) \subset Ann(L_e) \quad \forall\, w \in W$$

The other extreme is $I_o \equiv I_{w_o}$ which is the unique minimal primitive ideal of infinitesimal character ρ. (w_o is the longest element of W)

Let Π denote the simple roots of $\mathfrak{g}$. And for any $\alpha \in \Pi$, let

$$I_\alpha \equiv Ann(L_{-s_\alpha}) = Ann(M(-s_\alpha\rho)/M(-\rho))$$

The primitive ideals I_α are "almost" minimal. The correspond to the primitive ideals that immediately cover (in the sense of posets) the minimal ideal I_o

Theorem

The primitive ideals I_α, $\alpha \in \Pi$, are all distinct from each other and I_o. Any primitive ideal strictly containing I_o contains at least one of the I_α.

Definition

The **tau invariant** of a primitive ideal I containing I_o is

$$\{s \in \Pi \mid I_s \subset I\}$$

Of course, $Ann(\pi) = Ann(\pi')$ means their tau invariants must coincide.

Tau invariants of the annihilators of HC-modules, though a weaker invariant than annihilators themselves, can at least be computed directly by `atlas`.

A partitioning of HC cells

Recall that the W-graph of a block induces the following graph on a cell: for each element $i \in C$ we attach

- a vertex $v[i]$
- a tau invariant $\tau[i]$ = tau invariant of $Ann(\pi_i)$
- a list of edges with multiplicities $e[i] = [(j_1, m_1), (j_2, m_d), \ldots, (j_k, m_k)]$

Since $\tau[i] = \tau[j]$ is a necessary condition for $Ann(\pi_i) = Ann(\pi_j)$ it makes sense to group together those cell elements whose vertices have the same tau invariants.

Thus, we define a partitioning P_1 of C by grouping together those vertices with the same tau-invariant.

Call such a collection a P_1-*subcell* of C.

Next for element i in any particular P_1-subcell, attach the following *second order tau invariant*

$$\tau_2[i] = \{\tau[j] \mid j \in \text{edge vertices of } i\}$$

and say that two elements i, j belong to the same P_2-*subcell* if

$$\tau_2[i] = \tau_2[j] \quad .$$

Similarly, set

$$\tau_3[i] = \{\tau_2[j] \mid j \in \text{ edge vertices of } i\}$$

and say that two elements i, j belong to the same P_3-subcell if

$$\tau_3[i] = \tau_3[j] \quad .$$

Eventually, since there are only a finite number of cell elements this iterative partitioning scheme must stabilize. Let P_∞ denote the final stable partitioning (the first P_j for which $P_{j+1} = P_j$).

Lemma

The P_∞ partitioning of a cell of Harish-Chandra modules is compatible with the partitioning of the cell into subcells consisting of representations with the same primitive ideal. In fact,

$$Ann(\pi) = Ann(\pi') \Rightarrow \pi \text{ and } \pi' \text{ live in same } P_\infty\text{-subcell.}$$

Theorem

Let C be any cell in any real form of any exceptional group G. Then the P_∞ partitioning of C coincides precisely with the partitioning of the cell into sets irreducible Harish-Chandra modules sharing the same primitive ideal.

proof:

$$\#P_\infty\text{-subcells} = \text{dim special } W\text{-rep attached to cell} = max\#\text{primitive ideals in cell}$$

See summary table "Cells, Orbits, and Weyl group Representations"