HC-Cells, Nilpotent Orbits, Primitive Ideals, and Weyl Group Representations

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ABSTRACT: Let G be the real points of a connected linear reductive complex algebraic group defined over $\mathbb{R}$ and let $\widehat{G}_{adm,\lambda}$ be the set of equivalences classes of irreducible admissible representations of G of infinitesimal character λ, which we assume to be regular and integral. The Atlas software enumerates the representations in $\widehat{G}_{adm,\lambda}$, and computes the Kazhdan-Lusztig-Vogan polynomials $P_{x,y}(q)$ which not only prescribe the Jordan-Hölder decomposition of standard modules in terms of the irreducibles in $\widehat{G}_{\lambda,adm}$, but can also be used to endow the set $\widehat{G}_{adm,\lambda}$ with the structure of a W-graph, a certain weighted directed graph. The strongly connected components of this W-graph are HC-cells. In this talk I will describe how the induced W-graph structure of an HC-cell $\mathcal{C}$ allows one to compute the (common) associated variety of the annihilators of the representations in $\mathcal{C}$, and in fact, to determine when two representations $x, y \in \mathcal{C}$ share the same annihilator.

1. SETUP

Let G be the real points of a connected linear reductive complex algebraic group $\mathbb{G}$ defined over $\mathbb{R}$. We fix a Cartan involution θ of G and let $K = G^\theta$ be the corresponding maximal compact subgroup. θ extends to holomorphic involution of $\mathbb{G}$ and we set $\mathbb{K} = \mathbb{G}^\theta$.

Let $\widehat{G}_{adm}$ denote the set of (equivalence classes of) irreducible admissible representations of G and let $\mathcal{HC}$ denote the corresponding set of irreducible Harish-Chandra modules - obtained by sending an irreducible admissible representation $x \in \widehat{G}_{adm}$ to its space V_x of smooth K-finite vectors and then regarding the latter as an irreducible $(\mathfrak{g}, K)$-module.

In this talk we shall be concerned with explicitly attaching certain algebraic invariants to the representations in $\widehat{G}_{adm}$.

The most basic of these algebraic invariants is the infinitesimal character of a representation. By Schur's Lemmma, elements of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(g)$ act by scalars on any particular V_x, and these scalars are in turn prescribed by a certain Harish-Chandra homomorphism ϕ_{λ_x} which is, in turn, prescribed by a certain element of $\lambda_x \in \mathfrak{h}^*/W$; $\mathfrak{h}$ being a Cartan subalgebra of $\mathfrak{g}$ and W being the Weyl group of $\mathfrak{g}$. The element λ_x is called the *infinitesimal character* of x and one has

$$\widehat{G}_{adm} = \coprod_{\lambda \in \mathfrak{h}^*/W} \widehat{G}_{adm,\lambda}$$

$\widehat{G}_{adm,\lambda}$ being the set of irreducible admissible representations with infinitesimal character λ. Each $\widehat{G}_{adm,\lambda}$ is a finite set.

When λ is regular and integral (meaning $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{>0}$ for any simple coroot $\check{\alpha}$), the Atlas software can enumerate the irreducible representations in $\widehat{G}_{adm,\lambda}$. In fact, the Atlas software endows the set $\widehat{G}_{adm,\lambda}$ (λ again regular and integral) with a certain graph structure. The main point of this talk is to show how this graph structure can in turn be used to explicitly attach nilpotent orbits and primitive ideals to the

admissible representations in $\widehat{G}_{adm,\lambda}$. I should perhaps point out that it what follows it is most helpful to think of the representations in $\widehat{G}_{adm,\lambda}$ (as well as the corresponding Harish-Chandra modules $\mathcal{HC}$) as having been explicitly enumerated $0, 1, 2, \ldots, n$ (which is in fact exactly how the Atlas software enumurates $\widehat{G}_{adm,\lambda}$.

2. Blocks and Cells

Henceforth, we fix λ to be a regular infinitesimal character. As we have remarked above, the Atlas software can enumerate the irreducible admissible representations with infinitesimal character λ. In fact, the software does a lot more than simply count representations, it also endows the set $\widehat{G}_{adm,\lambda}$ with a certain directed, weighted, graph structure.

The way in which Atlas actually enumerates the representations in $\widehat{G}_{adm,\lambda}$ is described in detail in [Adams-du Cloux]. For our purposes here, we do not need to understand the connection between Atlas's enumeration and more conventional Langland's parameters; however, it is somewhat helpful to have a grasp of the basic combinatorial data that goes into the Atlas enumeration. As described in Adams-du Cloux, a (translation family of) irreducible admissible representation is corresponds to a certain pair (x, y) where x is a $\mathbb{K}$-orbit in $\mathbb{G}/\mathbb{B}$ and y is a $\mathbb{K}^\vee$-orbit in $\mathbb{G}^\vee/\mathbb{B}^\vee$; here $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$, $\mathbb{G}^\vee$ is the dual group of $\mathbb{G}$, $\mathbb{B}^\vee$ a Borel subgroup of $\mathbb{G}^\vee$, and $\mathbb{K}^\vee$ is the complexification of a real form of a maximal compact subgroup $K^\vee$ of a real form $G^\vee$ of $\mathbb{G}^\vee$.

The Atlas software enumerates the admissible representations in $G_{adm,\lambda}$ by first enumerating the $\mathbb{K}$-orbits in $\mathbb{G}/\mathbb{B}$ and then for each real form $G^\vee$ of the dual group $\mathbb{G}^\vee$ enumerating the $\mathbb{K}^\vee$–orbits in $\mathbb{G}^\vee/\mathbb{B}^\vee$ and checking a certain compatibility conditions. The collection of irreducible admissible representations arise in this fashion from a particular real form on the dual side constitute what is called a **block** of representations, and one has

$$\widehat{G}_{adm,\lambda} \approx \coprod_{\text{dual real forms } G^\vee} \{(x, y) \mid x \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B} \quad , \quad y \in \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee \quad , \quad x, y \text{ compatible}\}$$

We remark that in a similar fashion, the irreducible admissible representations of a real form $G^\vee$ of $\mathbb{G}^\vee$, can be broken up into blocks corresponding to various real forms of $\mathbb{G}$. In fact, Vogan duality tells us that if (x, y), $x \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B}$, $y \in \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee$ corresponds to an admissible representation of G, then (y, x) corresponds to an admissible representation of the real form $G^\vee$ of $\mathbb{G}^\vee$ corresponding to $\mathbb{K}^\vee$. At any rate, in the Atlas world, irreducible admissible representations come in blocks corresponding to pairs $(G, G^\vee)$.

Example 2.1. Let $\mathbb{G}$ be the simply connected complex linear group with Lie algebra E_8. We note that $\mathbb{G}$ is self-dual and has three real forms: the compact real form e_8, the quaterionic real form $E_8 (e_7 + \mathfrak{su}(2))$ and the split real form $E_8 (\mathbb{R})$. Below is a table showing the number of irreducible admissible representations (at regular infinitesimal character λ) in each block of $\mathbb{G}$

	$\mathfrak{e}_8$	$E_8 (e_7 + \mathfrak{su}(2))$	$E_8 (\mathbb{R})$
$\mathfrak{e}_8$	0	0	1
$E_8 (e_7 + \mathfrak{su}(2))$	0	3150	73410
$E_8 (\mathbb{R})$	1	73410	453060

The total number of irreducible admissible representations of $E_8 (\mathbb{R})$ with infinitesimal character λ is thus

$$1 + 73410 + 453060 = 526471 \quad .$$

Similarly, at any fixed regular integral infinitesimal character λ, there are $3250 + 73410 = 76\,660$ irreducible admissible representations of the quaterionic real form.

Each block of representation can in turn be partitioned into smaller subsets called cells. We describe in a minute how Atlas does this; but we'll start with a purely formal definition.

Definition 2.2. *Given two irreducible Harish-Chandra modules x, y in $\mathcal{HC}_\lambda$, we shall say that $x \rightsquigarrow y$ if there exists an irreducible finite-dimensional representation F of G such that F occurs in the tensor algebra of $\mathfrak{g}$ and x occurs as a subquotient of $y \otimes F$. We say that*

$$x \sim y$$

*if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The equivalence classes for the relation $\sim$ are called **cells** of Harish-Chandra modules, or HC-cells.*

Alternatively, one can define HC-cells by putting a certain graph structure on the set $\mathcal{HC}_\lambda$.

Definition 2.3. *Given $x, y \in \mathcal{HC}_\lambda$, we say that there is a (directed) edge $x \longrightarrow y$ from x to y whenever x occurs as a subquotient of $y \otimes \mathfrak{g}$.*

It is easy to see that the transitive closure of "$\longrightarrow$" replicates the relation "$\rightsquigarrow$" of Definition (??) and that the strongly connected components (that is the bidirectionally connected) of the graph with vertices $x \in \mathcal{HC}_\lambda$ and edges $x \longrightarrow y$ as defined above coincide with the HC-cells of $\mathcal{HC}_\lambda$. It also turns out that each connected component of this graph coincides with a particular block of representations. The Atlas software computes this graphical structure, block by block (connected component by connected component) essentially as a by-product its computation of the Kazhdan-Lusztig-Vogan (KLV-) polynomials for the various real forms of $\mathbb{G}$.

In fact, the Atlas software endows the set $\mathcal{HC}_\lambda$ with an even more elaborate graphical structure.

Definition 2.4. *Let B be a block of irr HC modules of inf char λ.*

*The W-**graph** of B is the weighted digraph where:*

- *the **vertices** are the elements $x \in B$*
- *there is an **edge** $x \to y$ of **multiplicity** m between two vertices if*

$$\text{coefficient of } q^{(|x|-|y|-1)/2} \text{ in } P_{x,y}(q) = m \neq 0$$

- *there is assigned to each vertex x a subset $\tau(x)$ of the set of simple roots of $\mathfrak{g}$, the **descent set** of x.*

Remark 2.5. This W-graph structure, although computed via KLV polynomials, is in fact a significant enhancement of the graph structure we defined before in purely representation theoreticl terms. In particular,

- The multiplicity of an edge $x \to y$ coincides with the multiplicity with which the representation x occurs in $y \otimes \mathfrak{g}$.
- The descent set of a vertex x coincides with the tau invariant of the annihilator of x in $U(\mathfrak{g})$.

Example 2.6. Let $G_2(\mathbb{R})$ be the split real form of the simply connected complex group with Lie algebra G_2. G_2 is self-dual, and the output of Atlas's `wgraph` command for the big block $G_2(\mathbb{R}) \times G_2(\mathbb{R})$ is block descent edge vertices, element set multiplicities 0   1 2 (3,1) 2 1 (4,1) 3 1 (0,1),(1,1),(6,1) 4 2 (0,1),(2,1),(5,1) 5 1 (4,1),(8,1) 6 2 (3,1),(7,1) 7 1 (6,1),(11,1) 8 2 (5,1),(10,1) 9 1,2 (7,1),(8,1) 10 1 (8,1) 11 2 (7,1)

This data leads to the following W-graph

$$\#0_{\{\}} \qquad \text{(LDS)}$$

$$\#1_{\{2\}} \underset{1}{\overset{1}{\rightleftarrows}} \#3_{\{1\}} \qquad\qquad \#4_{\{2\}} \underset{1}{\overset{1}{\rightleftarrows}} \#2_{\{1\}}$$

$$1 \Big\| 1 \qquad\qquad 1 \Big\| 1$$

$$\#6_{\{2\}} \qquad\qquad \#5_{\{1\}}$$

$$1 \Big\| 1 \qquad\qquad 1 \Big\| 1$$

$$\#11_{\{2\}} \underset{1}{\overset{1}{\rightleftarrows}} 7_{\{1\}} \qquad\qquad \#8_{\{2\}} \underset{1}{\overset{1}{\rightleftarrows}} \#10_{\{1\}}$$

$$\#9_{\{1,2\}} \qquad \text{(trivial)}$$
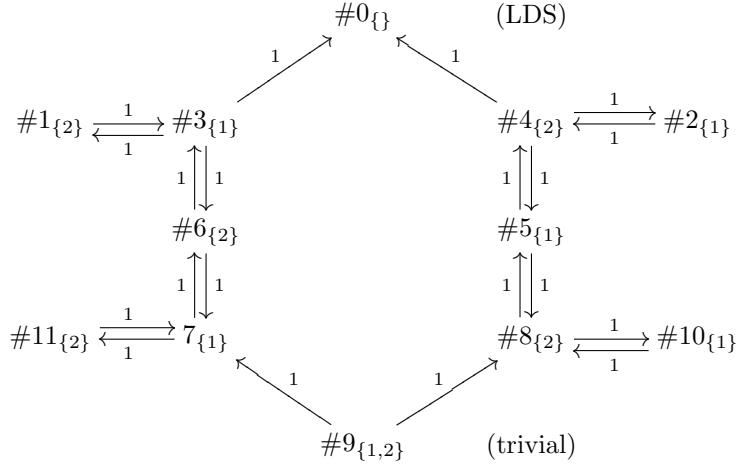
Note that there are four strongly connected components to this graph

$$\{\#0\} \quad , \quad \{\#1, \#3, \#6, \#7, \#11\} \quad , \quad \{\#2, \#4, \#5, \#8, \#10\} \quad , \quad \{\#9\}$$

and thus four HC-cells in this block. I might also point out, to help you get your bearings, that, at infinitesimal character ρ, vertex $\#0$ corresponds to the large discrete series representation while vertex $\#0$ corresponds to the trivial representation.

3. From W-graphs to invariants

I'll now describe how one can utilize the W-graph structure on $\widehat{G}_{adm,\lambda}$ to determine certain algebraic invariants for the representations in $\widehat{G}_{adm,\lambda}$. Before beginning though I would like to point out what to me seems most remarkable: the algebraic invariants attached to a vertex are not determined by properties of the vertex *per se* but rather by how the vertex is situated in W-graph of the block that contains it.

3.1. Primitive Ideals.
The invariants to be considered here will actually be invariants attached most directly to the primitive ideal in $U(\mathfrak{g})$. I point out, however, that there are other invariants (e.g. the associated variety of an irreducible Harish-Chandra module [16]) that are attached directly to Harish-Chandra modules themselves and which are in fact an invariant of an entire cell of Harish-Chandra modules and so are at least partially determined by the W-graph structure of the block in which a representation sits.

Definition 3.1. *Let V be an irreducible $U(\mathfrak{g})$-module.*

$$Ann(V) := \{X \in U(\mathfrak{g}) \mid Xv = 0 \quad , \quad \forall \ v \in V\}$$

*is a two-sided ideal in $U(\mathfrak{g})$. It is called the **primitive ideal** in $U(\mathfrak{g})$ attached to V.*

It is easy to see that $Ann(V) = Ann(V') \implies inf \ ch \ V = inf \ ch \ V'$ and so it makes sense to talk about the primitive ideals at infinitesimal character λ. We set

$$Prim(\mathfrak{g})_\lambda := \text{ set of primitive ideals in } U(\mathfrak{g}) \text{ with infinitesimal character } \lambda$$

For any $x \in \widehat{G}_{adm,\lambda}$, let V_x be the corresponding Harish-Chandra module and let $I_x := Ann(V_x)$. The correspondence

$$\phi : \widehat{G}_{adm,\lambda} \to Prim(\mathfrak{g})_\lambda : x \longmapsto I_x$$

is often one-to-one, but generally speaking, several-to-one. We obtain by this correspondence a fairly fine partitioning of $\widehat{G}_{adm,\lambda}$. Towards the end of this talk, we shall show how the W-graph structure of a block

allows one to determine exactly when two representations living in the same cell share a common primitive ideal.

3.2. **Nilpotent Orbits.** $U(\mathfrak{g})$ is naturally filtered according to
$$U^n(\mathfrak{g}) = \{X \in U(\mathfrak{g}) \mid X = \text{product of} \leq n \text{ elements of } \mathfrak{g}\}$$
and the corresponding graded algebra
$$gr(U(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} U^n(\mathfrak{g}) / U^{n-1}(\mathfrak{g})$$
is well defined, and, in fact,
$$gr(U(\mathfrak{g})) \approx S(\mathfrak{g}) := \text{the symmetric algebra of } \mathfrak{g}$$
$$\approx \mathbb{C}[\mathfrak{g}^*] := \text{the ring of polynomials on } \mathfrak{g}^*$$

Definition 3.2. *Let J be a primitive ideal and set*
$$\mathcal{V}(J) = \{\lambda \in \mathfrak{g}^* \mid \phi(\lambda) = 0 \quad \forall \phi \in gr(J)\}$$
The affine variety $\mathcal{V}(J)$ is called the associated variety *of J.*

Theorem 3.3 (Borho, Brylinski, Joseph)**.** *$\mathcal{V}(J)$ is a closed, G-invariant subset of $\mathfrak{g}^*$. In fact, $\mathcal{V}(J)$ is the Zariski closure of a single nilpotent orbit in $\mathfrak{g}^*$*

Definition 3.4. *Let $x \in HC_\lambda$ .The* **nilpotent orbit attached to** *x is the unique dense orbit $\mathcal{O}_x$ in $\mathcal{V}(Ann(x))$.*

Lemma 3.5. *If $x, y \in \widehat{G}_{adm,\lambda}$ belong to the same HC-cell, then $\mathcal{O}_x = \mathcal{O}_y$.*

This is so because, roughly speaking, tensoring with $\mathfrak{g}$ only affects the contribution of lower order terms that are anyway annihilated by the gradation process. We remark though that different cells can share the same nilpotent orbit. We will describe below how one can utilize the W-graph structure of the cell to explicitly identify which nilpotent orbit is attached to the representations of cell.

4. Weyl group representations

One of the most remarkable developments in 1980's was the discovery of how the irreducible representations of Weyl group mediate the myriad of connections between representation theory, the classification theory of primitive ideals, nilpotent orbits. It should be no surprise then that the W-graph structure of a block should have something to do with Weyl group representations.

In fact, the Weyl group W, or rather the Iwasora-Hecke deformation of W, is the fundamental apparatus at play in the construction of the KLV-polynomials, which in turn allow one to explicity compute the W-graph structure of $\widehat{G}_{adm,\lambda}$. Indeed, it turns out that the Weyl group action on the free $\mathbb{Z}$-module $\mathbb{Z}\widehat{G}_{adm,\lambda}$ obtained by specializing the action of the Hecke algebra to $q = 1$ identifies with coherent continuation representation [15] of W on the Grothendieck group of $G_{adm,\lambda}$. This action induces a particular (in general reducible) W-representation on each cell which in turn has a unique *special* constituent σ_C occuring with multiplicity one. Moreover, it turns out that each of the following means of attaching a W-representation to a cell C leads to the same result:

(i) $C \longrightarrow$ cell representation $\xrightarrow{\textit{unique special constituent}} \sigma_C \in \widehat{W}$

(ii) $C \longrightarrow$ nilpotent orbit $\mathcal{O}_C \longrightarrow$ a nilpotent orbit $\mathcal{O}_C$ and a trivial local system on $\mathcal{O}_C$
$\xrightarrow{\textit{Springer correspondence}} \sigma_C \in \widehat{W}$

(iii) $C \longrightarrow \{\text{primitive ideal } I_x \mid x \in C\} \rightarrow W \cdot span_{\mathbb{C}} \{\text{Goldie rank polynomial } p_{I_x} \mid x \in C\} \longrightarrow \sigma_C \in \widehat{W}$

So it would seem that the ultimate arbitrator in all that follows is again the Weyl group. What might be a bit surprising is how easy it is to obtain the representation of W carried by a cell from its induced W-graph. Let $\mathcal{C}$ be the W-graph of a cell (obtained by restricting the W-graph of $\widehat{G}_{adm,\lambda}$ to one of its strongly connected components). Regarding the vertices of $\mathcal{C}$ as a basis for the free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{C}$ we define an action of the reflection corresponding to a simple root α on $\mathbb{Z}\mathcal{C}$ by

$$T_\alpha x = \begin{cases} -x & \alpha \in \tau(x) \\ x + \sum_{y:\alpha\in\tau(y)} m_{y\to x}y & \alpha \notin \tau(x) \end{cases}$$

Here $m_{y\to x}y$ is the multiplicity of the edge $y \to x$ and $\tau(x)$ is the descent set of x. It turns out that these operators satisfy $T_\alpha \circ T_{alpha} = 1$ as well as the braid relations for the simple generators of W and so yield a representation of the Weyl group on $\mathbb{Z}\mathcal{C}$. As remarked above this representation coincides with the W-representation on the cell induced by the coherent continuation representation on the Grothendieck group of $G_{adm,\lambda}$. It is possible, using the above formula, to explicitly compute the trace of the action of a representative of each conjugacy class in W, thereby determine the character of the cell representation, and then to identify explicitly the representation of W carried by the cell by expressing this character as a sum of irreducibles. In fact, this simple method of computation is tractable even for the large cells of E_8 (which contain roughly 50,000 vertices). In an appendix to this lecture we provide a set of tables showing the cell representations so obtained for all the exceptional groups.

5. Attaching Nilpotent Orbits to Cells

Computing the cell representation, identifying the special constituent σ_C of the cell representation and then applying the inverse Springer correspondence is one way of attaching a nilpotent orbit to a cell. In this section we provide an alternative method that utilizes only the properties of special nilpotent orbits and the combinatorial properties of the W-graph of a cell.

We first note that having restricted attention to representations with regular integral infinitesimal character, only a special class of nilpotent orbits will arise as the associated varieties of annihilators of these representations. This class of nilpotent orbits (which are in fact called *special* nilpotent orbits can be characterized in several different ways (see Definition/Theorem ?? below).

Let $\mathcal{N}$ be the nilpotent cone of nilpotent elements in $\mathfrak{g}$ and let G_{ad} denote the (complex) adjoint group of $\mathfrak{g}$. We shall denote by $G_{ad}/\mathcal{N}$ the set of G_{ad}-orbits in $\mathcal{N}$ and we identify this set with the set of nilpotent orbits in $\mathfrak{g}^*$ via some fixed invariant non-degenerate bilinear on $\mathfrak{g}$). We note that the set $G_{ad}/\mathcal{N}$ carries a natural partial ordering

$$\mathcal{O}' \leq \mathcal{O} \iff \mathcal{O}' \subset \overline{\mathcal{O}} := \text{the Zariski closure of } \mathcal{O}$$

In [13], and a bit more generally, in [5] a *duality map* $d : G_{ad}/\mathcal{N} \to G_{ad}/\mathcal{N}$ is defined. This map has the following properties:

- d is order-reversing: $\mathcal{O}' \leq \mathcal{O} \iff d(\mathcal{O}) \leq (\mathcal{O}')$.
- d restricted to its image is an involution: $d \circ d \circ d = d$.

We note that for classical $\mathfrak{g}$, where the nilpotent orbits are parameterized by certain partitions of $n = $ dimension of the standard representation of $\mathfrak{g}$, the duality map restricted to its image corresponds to the operation that sends an orbit corresponding to partition $\mathbf{p}$ to the orbit corresponding to the transpose partition $\mathbf{p}^t$.

Definition/Theorem 5.1. *A nilpotent orbit $\mathcal{O}$ is **special** if any of the following (equivalent) properties holds:*

- *$\mathcal{O}$ is the dense orbit in the associated variety of a primitive ideal with regular integral infinitesimal character.*

- $\mathcal{O}$ lies in the image of the Spaltenstein-Barbasch-Vogan duality map d.
- The irreducible representation of W corresponding to $\mathcal{O}$ and the trivial local system on $\mathcal{O}$ via the Springer correspondence is a special representation of the Weyl group in the sense of [Lustig].

As noted above, because we have restricted attention to representations with regular integral infinitesimal character, the nilpotent orbit $\mathcal{O}_C$ attached to a cell C will be always be special nilpotent orbit. In order to explicitly identify the special nilpotent orbit attached to a particular cell we shall need to make use of the partial ordering of the special orbits, the duality map, and a particular subset of the special orbits; the Richardson orbits.

5.1. **Standard Levis and Richardson orbits.** Let Γ be a subset of the simple roots Π of $\mathfrak{d}\,(\mathfrak{g};\mathfrak{h})$. The corresponding *standard Levi subalgebra* $\mathfrak{l}_\Gamma$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_\alpha$, $\alpha \in \Gamma$, and the Cartan subalgebra $\mathfrak{h}$. Equivalently,

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \mathbb{Z}\Gamma} \mathfrak{g}_\alpha$$

Definition/Theorem 5.2. *Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$ and nilradical $\mathfrak{n}$, and let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. The nilpotent orbit **induced from** $\mathcal{O}_\mathfrak{l}$ is*

$$ind_\mathfrak{l}^\mathfrak{g}\,(\mathcal{O}_\mathfrak{l}) := \;\; unique\;\; dense\;\; orbit\;\; in\;\; G_{ad} \cdot (\mathcal{O}_\mathfrak{l} + \mathfrak{n})$$

*(The theorem part of this statement is that the orbit $ind_\mathfrak{l}^\mathfrak{g}\,(\mathcal{O}_\mathfrak{l})$ specified as above is well-defined.) When $\mathcal{O}_\mathfrak{l} = \mathbf{0}_\mathfrak{l}$ (the trivial orbit in $\mathfrak{l}$) then such an induced orbit is called a **Richardson orbit**. When $\mathfrak{l} = \mathfrak{l}_\Gamma$ is a standard Levi subalgebra and $\mathcal{O}_\mathfrak{l} = \mathbf{0}_{\mathfrak{l}_\Gamma}$, then $ind_{\mathfrak{l}_\Gamma}^\mathfrak{g}\,(\mathbf{0}_{\mathfrak{l}_\Gamma})$ will be called the **Richardson orbit corresponding to** Γ and denoted by $\mathcal{R}_\Gamma$.*

The trivial orbit is always special (in fact, the trivial orbit is always the dual of the principal nilpotent orbit and vice-versa). Moreover, by a theorem of Lusztig, induction preserves the property of "special-ness". And so Richardson orbits are always special nilpotent orbits. However, not all special orbits are Richardson. (We note, nevertheless, that for $\mathfrak{sl}\,(n,\mathbb{C})$ every nilpotent orbit is both special and Richardson, and that in general the existence of non-Richardson special orbits is related to the lack of injectivity of the duality map $d : G_{ad}\backslash\mathcal{N} \to G_{ad}\backslash\mathcal{N}$).

5.2. **The Spaltenstein-Vogan criterion.** We next note that from the explicit formula for the cell representation (cf. §5) that for any $x \in C$ the cell representation will always contain the sign representation of $W_{\tau(x)}$. Using this fact and the following result of Spaltenstein

Proposition 5.3 (Corollary 1 of [12]). *A special orbit $\mathcal{O}$ is contained in the closure of a Richardson orbit $\mathcal{R}_\Gamma$ if and only if the (special) W-representation attached to $(\mathcal{O}, \mathbf{1})$ contains the sign representation of W_Γ. Here W_Γ is the reflection subgroup of W generated by the simple reflections s_α, $\alpha \in \Gamma$.*

David Vogan has proved the following criterion

Theorem 5.4. *Suppose C is a cell of Harish-Chandra modules and let $\mathcal{O}_C$ be the associated nilpotent orbit (cf. §4.2). Then*

$$\mathcal{O}_C \subset \overline{\mathcal{R}_-} \quad \textbf{iff} \quad \exists\, x \in C\; s.t.\; \Gamma \subset \tau(x)$$

Thus, the descent sets that occur amongst the representations in a cell C constain which Richardson orbits can contain $\mathcal{O}_C$ in their closures and vice-versa. However, because not every special orbit is Richardson, knowing which Richardson orbits contain a given cell orbit $\mathcal{O}$ in their closures is not sufficient to determine $\mathcal{O}$. However, the following is true:

Observation 5.5 (Vogan). *Every special nilpotent orbit $\mathcal{O}$ is determined by the Richardson orbits $\mathcal{R}_\Gamma$ that contain $\mathcal{O}$ in their closures together with the Richardson orbits that contain $d(\mathcal{O})$ in their closures.*

5.3. Tau signatures. I will now describe how the Spaltenstein-Vogan criterion and Vogan's observation can be put to work to explicity attach nilpotent orbits to HC-cells.

First of all, the correspondence $\Pi \to G_{ad}\backslash\mathcal{N} : \Gamma \mapsto \mathcal{R}_\Gamma$ is hardly one-to-one. But by a classical result of Dynkin, conjugacy classes of Levi subalgebras in a semisimple Lie algebra $\mathfrak{g}$ (and hence Richardson orbits in $\mathfrak{g}$) are in a one-to-one correspondence with W-conjugacy classes in 2^Π, the set of subsets of Π. This motivates the following definition:

Definition 5.6. *Fix an ordering of the simple roots Π and let $\Psi \subset 2^\Pi$ be the collection*

$$\Psi = \{\Gamma \in \Pi \mid \Gamma \text{ is the first in the lexicographical ordering of its } W\text{-conjugacy class}\}$$

*We call such Ψ a set of **standard** Γ's.*

By Dynkin's result any such Ψ is in a one-to-one correspondence with the set of Richardson orbits, and this in turn allows us to put the following partial order on Ψ:

$$\Gamma \leq \Gamma' \quad \Longleftrightarrow \quad \mathcal{R}_\Gamma \subseteq \overline{\mathcal{R}_{\Gamma'}}$$

Definition 5.7. *Let $\mathcal{O}$ be a special nilpotent orbit. The **tau signature** of $\mathcal{O}$ is the pair of subsets of Ψ defined by*

$$\tau_{sig}(\mathcal{O}) = \left(\min\left\{\Gamma \in \Psi \mid \mathcal{O} \subset \overline{\mathcal{R}_\Gamma}\right\} \ , \ \min\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{\mathcal{R}_\Gamma}\right\}\right)$$

It follows from Vogan's observation that a special orbit is completely determined by its tau signature.

Next,

Definition 5.8. *Let C be an HC-cell and set*

$$\tau(C) = \{\tau(x) \mid x \in C\}$$
$$\tau^\vee(C) = \{\Pi - \tau(x) \mid x \in C\}$$

*The **tau signature** of C is the pair of subsets of Ψ defined by*

$$\tau_{sig}(C) = \{\min(\tau(C) \cap \Psi) \ , \ \min(\tau^\vee(C) \cap \Psi)\}$$

In terms of these tau signatures, we now have the following criterion:

Criterion 5.9. *Let C be an HC-cell and let $\mathcal{O}$ be a special nilpotent orbit. Then*

$$\mathcal{O}_C = \mathcal{O} \quad \Longleftrightarrow \quad \tau_{sig}(C) = \tau(\mathcal{O}) \ .$$

(That $\tau^\vee(C)$ is the correct counterpart to $\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{\mathcal{R}_\Gamma}\right\}$ comes from the fact that the $\tau(C)$ is a collection of descent sets for an HC-cell for a block of $\mathbb{G}$ then $\tau^\vee(C)$ will be a collection of descent sets for a cell in a block of $\mathbb{G}^\vee$ and the fact that this correspondence is consistent with the Barbasch-Vogan duality map $d : G_{ad}\backslash\mathcal{N}_\mathfrak{g} \to G_{ad}^\vee\backslash\mathcal{N}_{\mathfrak{g}^\vee}$).

Let me now give an example showing how easy it is to apply this criterion.

Below is the Hasse diagram for the special orbits of $D_5 \approx \mathfrak{so}(10, \mathbb{C})$, with the orbits labeled by the corresponding partitions of 10. The dotted lines indicate the action of the duality map d:

The Special Orbits of D_5

$\mathcal{O}_{[9,1]}$

$\mathcal{O}_{[7,3]}$

$\mathcal{O}_{[5^2]}$ $\qquad\qquad$ $\mathcal{O}_{[7,1^3]}$

$\mathcal{O}_{[5,3,1,1]}$

$\mathcal{O}_{[4^2,1^2]}$

$\mathcal{O}_{[3^3,1]}$ $\qquad\qquad$ $\mathcal{O}_{[5,1^5]}$

$\mathcal{O}_{[3^2,2^2]}$

$\mathcal{O}_{[3^2,1^4]}$

$\mathcal{O}_{[2^4,1^2]}$ $\qquad\qquad$ $\mathcal{O}_{[3,1^7]}$

$\mathcal{O}_{[2^2,1^6]}$

$\mathcal{O}_{[1^{10}]}$
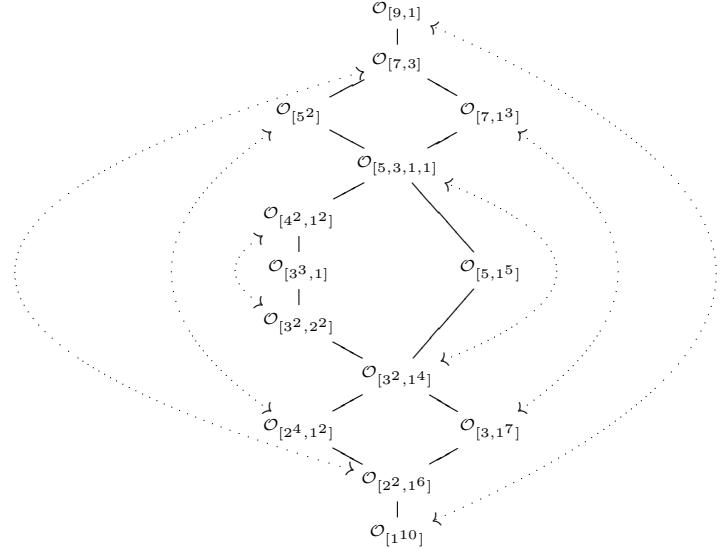
Below is the same diagram, except that we explicitly identify which of these special orbits is Richardson.

Richardson Orbits of D_5

$\mathcal{O}_{[9,1]} = \mathcal{R}_{\{\}}$

$\mathcal{O}_{[7,3]} = \mathcal{R}_{\{1\}}$

$\mathcal{O}_{[5^2]} = \mathcal{R}_{\{1,3\}}$ $\qquad\qquad$ $\mathcal{O}_{[7,1^3]}$

$\mathcal{O}_{[5,3,1^2]} = \mathcal{R}_{\{1,4,5\}}$

$\mathcal{O}_{[4^2,1^2]} = \mathcal{R}_{\{1,2,4\}}$

$\mathcal{O}_{[3^3,1]} = \mathcal{R}_{\{1,2,4,5\}}$ $\qquad\qquad$ $\mathcal{O}_{[5,1^5]}$

$\mathcal{O}_{[3^2,2^2]}$

$\mathcal{O}_{[3^2,1^4]} = \mathcal{R}_{\{1,3,4,5\}}$

$\mathcal{O}_{[2^4,1^2]} = \mathcal{R}_{\{1,2,3,4\}}$ $\qquad\qquad$ $\mathcal{O}_{[3,1^7]} = \mathcal{R}_{\{2,3,4,5\}}$

$\mathcal{O}_{[2^2,1^6]}$

$\mathcal{O}_{[1^{10}]} = \mathcal{R}_{\{1,2,3,4,5\}}$
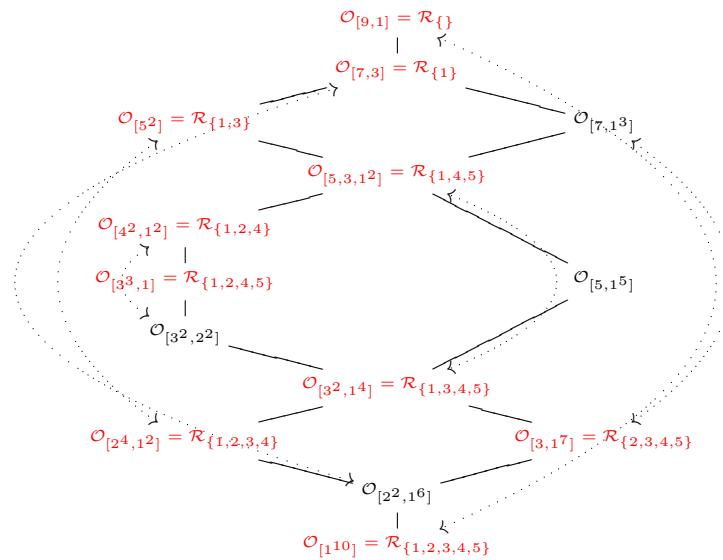
From the last diagram, we can easily read off the minimal Richardson orbits that contain a given special orbit, which Richardsonson orbits contain the dual of that orbit, and then use that information to ascribe tau signatures to the special orbits. This yields

Tau Signatures of Special Orbits of D_5
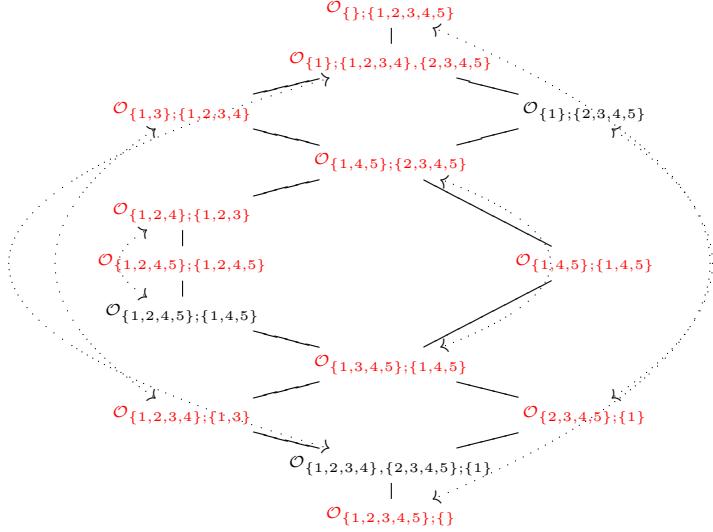
Next, we use the Atlas program to compute the W-graph of the big $(SO(5,5) \times SO(5,5))$ block of D_5. This yields a table like

```
// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {2} --> 1,2
1[3]: {1} --> 0
2[5]: {3} --> 0,3,4
3[13]: {5} --> 2
4[14]: {4} --> 2
  *
  *
  *
// cell #29:
0[328]: {1,2,4,5} --> 2,3
1[340]: {2,3,4,5} --> 2
2[358]: {1,3,4,5} --> 0,1
3[364]: {1,2,3} --> 0

// cell #30:
0[353]: {1,2,3,4,5}

// cell #31:
0[357]: {1,2,3,4,5}
```

From this table, and the partial ordering of the standard Γ's (from Figure) we can readily identify the tau signature of each cell of the block.

```
cell #     tau signature
    0      {} , {1,2,3,4,5}
    1      {1} , {1,2,3,4}
    2      {1} , {2,3,4,5}
    3      {1,3} , {1,3,4,5}
    *      *
    *      *
    *      *
   28      {2,3,4,5} , {1}
   29      {2,3,4,5} , {1}
   30      {1,2,3,4,5} , {}
   31      {1,2,3,4,5} , {}
```

Comparing the tau signatures of the special orbits with those of the cells we obtain the following cell-orbit correspondences

Cell-Orbit Correspondences for $SO(5,5)$

$$
\begin{array}{ccc}
\mathcal{O}_{[9,1]} & \quad & \#0 \\
\mathcal{O}_{[7,3]} & \quad & \#1 \\
\mathcal{O}_{[5^2]} \quad \mathcal{O}_{[7,1^3]} & \quad & \#2,5 \quad \#3,4 \\
\mathcal{O}_{[5,3,1^2]} & \quad & \#6,7,15,17 \\
\mathcal{O}_{[4^2,1^2]} & \quad & \#8,9 \\
\mathcal{O}_{[3^3,1]} \quad \mathcal{O}_{[5,1^5]} & \quad & \#10,12,13 \quad \#11,21 \\
\mathcal{O}_{[3^2,2^2]} & \quad & \#16,18 \\
\mathcal{O}_{[3^2,1^4]} & \quad & \#14,19,20,22 \\
\mathcal{O}_{[2^4,1^2]} \quad \mathcal{O}_{[3,1^7]} & \quad & \#24,25 \quad \#23,28,29 \\
\mathcal{O}_{[2^2,1^6]} & \quad & \#26,27 \\
\mathcal{O}_{[1^{10}]} & \quad & \#30,31
\end{array}
$$
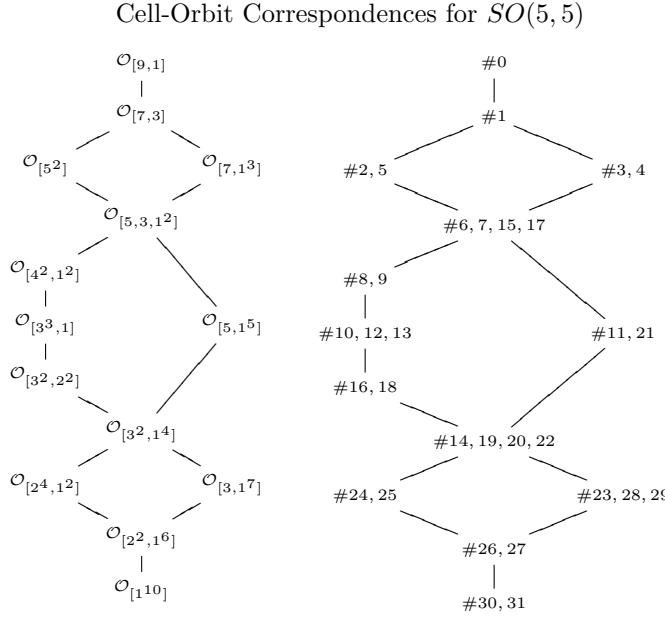
Note that several HC-cells may correspond to the same nilpotent orbit. This means, in particular, that annihilators of each of the representations in cells share the same associated variety. In this fashion, Atlas provides not only a means for enumerating the representations in $\widehat{G}_{adm,\lambda}$, but also means to explicitly attach nilpotent orbits to these representations.

6. Partitioning $\widehat{G}_{adm,\lambda}$ into subsets sharing the same primitive ideal

In the preceding section, we showed how the W-graph of a cell, in fact, just the set of vertex weights, allowed us to figure out which nilpotent orbit should be attached to the representations in the cell. In this section I shall so how the W-graph structure allows us to determine when two representations share the same primitive ideal.

First, however, I will quickly review some apparatus from of the theory of primitive ideals.

6.1. **Organization of $Prim(\mathfrak{g})_\lambda$.** Irreducible Harish-Chandra modules are the $U(\mathfrak{g}$-modules that most naturally in the study of admissible representations of a real reductive group. However, the most convenient setting for discussing primitive ideals is in the category of highest weight modules. We begin by recalling a basic construction of the simple highest weight modules.

Definition 6.1. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$ and let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \alpha$. For any $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ denote the Verma module of highest weight $\lambda - \rho$; that is to say, $M(\lambda)$ is the left $U(\mathfrak{g})$-module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda - \rho}$$

Theorem 6.2. Let $\lambda \in \mathfrak{h}^*$. Then

(i) The Verma module $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$ which is of highest weight $\lambda - \rho$.

(ii) Every irreducible highest weight module is isomorphic to some $L(\lambda)$.

Theorem 6.3 (Duflo). For $w \in W(\mathfrak{g},\mathfrak{h})$ let

$$L_w = \text{unique irreducible quotient of } M(-w\rho)$$

Then
$$\varphi : W \to Prim\,(\mathfrak{g})_\rho : w \to Ann\,(L_w)$$
is a surjection.

(Recall that $Prim\,(\mathfrak{g})_\rho$ is the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character ρ.)

In view of Duflo's theorem, parameterizing $Prim\,(\mathfrak{g})_\rho$ is tantamount to understanding the fiber of the map $\varphi : W \to Prim\,(\mathfrak{g})_\rho$. We have a special name for these fibers:

Definition 6.4. *Let $\sim$ be the equivalence relation on W defined by*
$$w \sim w' \iff Ann(L_w) = Ann(L_{w'})$$
*The corresponding equivalence classes of elements of W are called **left cells** in W.*

We can also collect together the elements of W which lead to the same nilpotent orbit. Before doing this formally, we note that because the simple highest weight modules L_w all have infinitesimal character ρ (actually $W \cdot \rho$) which is always regular and integral, the corresponding nilpotent orbits will always be special nilpotent orbits, via a theorem of Barbasch and Vogan ([5]). Let $\mathcal{S}$ denote the set of special nilpotent orbits. It turns out that the map $W \to \mathcal{S}$ given by

$$w \mapsto \mathcal{O}_w \equiv \text{the unique dense orbit in the associated variety of } Ann\,(L_w)$$

is also onto.

Definition 6.5. *Let $\approx$ be the equivalence relation on W defined by*
$$w \approx w' \iff \mathcal{O}_{Ann(L_w)} = \mathcal{O}_{Ann(L_{w'})}$$
*The corresponding equivalence classes of elements of W are **double cells** in W. For any special orbit $\mathcal{O} \in \mathcal{S}$ we denote by $\mathcal{C}_\mathcal{O}$ the corresponding double cell in W.*

Theorem 6.6 (Barbasch-Vogan, Joseph)**.**

(i) *The decomposition of W into left cells is refinement of its partitioning into double cells so that*

$$W = \coprod_{\mathcal{O} \in \mathcal{S}} \left(\coprod_{\text{left cells } \ell \subset \mathcal{C}_\mathcal{O}} \ell \right)$$

(ii) *Let $\mathcal{O}$ be a special orbit and let $\sigma_\mathcal{O}$ be the corresponding special representation of the Weyl group provided by the Springer correspondence. Then the number of left cells in $\mathcal{C}_\mathcal{O}$ is exactly the same as the dimension of $\sigma_\mathcal{O}$.*

In particular,

$$\left| Prim\,(\mathfrak{g})_\rho \right| = \sum_{\mathcal{O} \in \mathcal{S}} \dim \sigma_\mathcal{O}$$

We remark that via the Borho-Janzten translation principle this description of $Prim\,(\mathfrak{g})_\rho$ in terms of special nilpotent orbits and the corresponding special representations of W carries over, in exactly the same way, for $Prim\,(\mathfrak{g})_\lambda$ whenever λ is regular and integral.

6.2. **Two pictures at inf char ρ.** The above discussion can be summarized by following diagram HW-modules

W	$\{L_w \mid w \in W\}$	same inf char
$\cup$	$\cup$	
$\mathcal{C} : dbl\ cell$	$\{L_w \mid w \in \mathcal{C}\}$	same nilpotent orbit
$\cup$	$\cup$	
$\ell : left\ cell$	$\{L_w \mid w \in \ell\}$	same primitive ideal

Regarding W simply as a parameter set for the set of simple highest weight modules of infinitesimal character ρ, what we seek to develop next is an analogous picture for the set of irreducible admissible representations in a block of $\widehat{G}_{adm,\rho}$ as indexed by Atlas's block command.

$$
\begin{array}{ccc}
B : \text{ block of HC-modules} & \{\pi_x \mid x \in B\} & \text{same inf char} \\
\cup & \cup & \\
C : \text{cell of HC-modules} & \{\pi_x \mid x \in C\} & \text{same nilpotent orbit} \\
\cup & \cup & \\
? & \{\pi_x \mid x \in ?\} & \text{same primitive ideal}
\end{array}
$$

We shall show next how the W-graph structure of a cell enables us to figure out the analog of a left cell sitting inside an HC-cell. To do that we need to explain first a particular invariant of primitive ideals and its relation to the descent sets of the representations in a cell.

6.3. **Tau invariants.** Let L_w again denote the simple highest weight module of highest weight $-w\rho - \rho$ and let $I_w := Ann\,(L_w)$. We have

- I_{w_o} : unique max ideal (augmentation ideal, annihilator of triv rep)
- $I_e =$ unique min PI at inf char ρ ($\leq$ by inclusion)
- $I_{s_\alpha},\, \alpha \in \Pi$: "pen-minimal" ideals

A little more precisely, by this last fact we mean,

Theorem 6.7. *The primitive ideals I_{s_α}, $\alpha \in \Pi$, are all distinct from each other and I_e. Any primitive ideal strictly containing I_e contains at least one of the I_{s_α}.*

which in turn leads us to the following definition:

Definition 6.8. *The **tau invariant** of a primitive ideal I containing I_e is*
$$
\tau(I) = \{\alpha \in \Pi \mid I_{s_\alpha} \subset I\}
$$

The following theorem is the key to what follows. (It also justifies our notation for descent sets).

Theorem 6.9 (Vogan). *Let x be an element of a cell C of HC modules and let $\tau(x)$ be its descent set (as, for example, obtained from the W-graph of C). Then*
$$
\tau(x) = \textit{tau-invariant of } Ann(x)
$$

6.4. **A partitioning of HC cells.** Recall the the W-graph of cell C attaches to each for each irreducible admissible representation $x \in C$

- a vertex index i
- a tau invariant $\tau\,[i] =$ tau invariant of $Ann\,(x)$
- a list of edges with multiplicities $e\,[i] = [(j_1, m_1)\,, (j_2, m_d)\,, \ldots, (j_k, m_k)]$

With the goal of figuring out which irreducible admissible representations share the same primitive ideal, the first thing one might do is collect together the cell elements which share the same tau invariant (descent set). We shall call such collections τ_0-**subcells** and write

$$
x \sim_{\tau_0} y \quad \Longleftrightarrow \quad \tau(x) = \tau(y)
$$
$$
C = \coprod_{[x]_0 \in C/\sim_{\tau_0}} [x]_0
$$

We note that a τ_0-subcell corresponds a collection of admissible representations that share the same nilpotent orbit $\mathcal{O}_C$ and a common tau invariant.

Next we define τ_1-**subcells** by setting $\tau_1(x) = \{\tau(y) \mid x \to y \text{ is an edge}\}$ and writing

$$x \sim_{\tau_1} y \quad \Longleftrightarrow \quad \tau(x) = \tau(y) \text{ and } \tau_1(x) = \tau_1(y)$$

$$C = \coprod_{[x]_1 \in C/\sim_{\tau_1}} [x]_1$$

Similarly, $\boldsymbol{\tau_2}$ **subcells** are defined by setting $\tau_2(x) = \{\tau_1(y) \mid x \to y \text{ is an edge}\}$

$$x \sim_{\tau_2} y \quad \Longleftrightarrow \quad \tau_0(x) = \tau_0(y) \ , \ \tau_1(x) = \tau_1(y) \ , \text{ and } \tau_2(x) = \tau_2(y)$$

$$C = \coprod_{[x]_2 \in C/\sim_{\tau_2}} [x]_2$$

We can clearly continue in this fashion, obtaining at each new iteration a refinement of the previous partitioning of the cell. Since the cells are finite sets, however, eventually this iterative partitioning scheme must stabilize with τ_j-subcells coinciding with τ_{j+1}-subcells for all sufficiently large j. We shall refer to the ultimate stable partition of the cell as its τ_∞-partitioning and write

$$C = \coprod_{[x]_\infty \in C/\sim_{\tau_\infty}} [x]_{\tau_\infty}$$

Lemma 6.10. *The τ_∞ partitioning of a cell of HC-modules is compatible with the partitioning of the cell into subcells consisting of representations with the same primitive ideal:*

$$Ann(x) = Ann(y) \quad \Longrightarrow \quad x \text{ and } y \text{ live in same } \tau_\infty\text{-subcell.}$$

This follows from well-definedness of Translation Functor for primitive ideals.

Theorem 6.11. *Let C be any cell in any real form of any exceptional group G. Then the τ_∞ partitioning of C coincides precisely with the partitioning of the cell into sets of irreducible admissible representations sharing the same primitive ideal:*

$$x \sim_\infty y \quad \Longleftrightarrow \quad Ann(x) = Ann(y)$$

The proof of this theorem is by direct computation. Using the Atlas software we have explicitly computed the W-graph every such cell. It is then a simple matter to write a computer program that uses the W-graph data of a cell to partition it into a collection τ_∞-subcells. What one finds is that in every case

$$\#P_\infty\text{-subcells} = \dim \text{ special } W\text{-rep attached to cell}$$

But the dimension of the special representation attached to a cell is also the number of distinct primitive ideals with associated nilpotent orbit $\mathcal{O}_C$. Since the τ_∞-partitioning scheme is compatible with the partitioning by common primitive ideals, and so is at worse a coarsening of the partitioning by primitive ideals, we must conclude the τ_∞-partitioning scheme coincides with the partitioning by primitive ideals; as

$$\#P_\infty\text{-subcells} = max\#\text{primitive ideals in cell}$$

7. Some final remarks

What I think is most interesting about these results is the non-locality of the way in which algebraic invariants admissible representations are obtained from W-graph data. For example, our method of attaching a nilpotent orbit to an admissible representation x did not really use a property of corresponding vertex *per se*, but rather a property of its strongly connected component in the W-graph of $\widehat{G}_{adm,\lambda}$.

This non-locality is even more evident in the τ_∞ partitioning scheme. We first note that what this scheme is actually doing is collecting together vertices which encounter the same sets of tau invariants as moves $0, 1, 2, 3, \ldots$, vertices away along edges. What is even more curious about this scheme is that the vertices that share same τ_∞-invariant, and so the same primitive ideal, tend to be rather distant from each other in the cell's W-graph.

8. Acknowledgements

9. Sources

Information about the Atlas project, expository notes, and the Atlas software can be found on the Atlas for Lie Groups web site: http:\\atlas.math.umd.edu

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Appendix A. Cells, Orbits, Primitive Ideals and W-Representations

Let G be a real form of a linear connected complex algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$, and let $\widehat{G}_{adm,\lambda}$ be the set of (equivalence classes of) irreducible admissible representation of G of infinitesimal character λ, which we assume to be regular and integral. One can endow $\widehat{G}_{adm,\lambda}$ with the structure of a W-graph $\mathcal{G}_\lambda$ as follows:

- the **vertices** of $\mathcal{G}$ correspond to irreducible admissible representations in $\widehat{G}_{adm,\lambda}$
- Two vertices $x, y \in \widehat{G}_{adm,\lambda}$ are linked by a (directed) **edge** $x \to y$ if x occurs as a subquotient of $y \otimes \mathfrak{g}$.
- The **multiplicity** of an edge $x \to y$ is equal to the multiplicity with which x occurs as a subquotient of $y \otimes \mathfrak{g}$
- The τ-**invariant** of a vertex x is equal to the τ-invariant of $Ann_{U(\mathfrak{g})}(x)$ (that is, the tau invariant of the annihilator in $U(\mathfrak{g})$ of the irreducible $(\mathfrak{g}, K)$-module corresponding to $x \in \widehat{G}_{adm,\lambda}$, a subset of the simple roots of $\mathfrak{g}$).

The connected components of the W-graph $\mathcal{G}_\lambda$ corresponded to certain subsets of $\widehat{G}_{adm,\lambda}$ known as **blocks** of (irreducible admissible) representations. Blocks in $\widehat{G}_{adm,\lambda}$ are effectively parameterized by the set of strong real forms of the complex algebraic group $(G_{\mathbb{C}})^\vee$ dual to $G_{\mathbb{C}}$ corresponding to λ (albeit, some of these blocks may be empty).

The strongly connected components of the W-graph $\mathcal{G}_\lambda$, that is, the maximal sets of vertices that can be connected bidirectionally by sequences of directed edges are known as **cells**.

The Atlas software computes the W-graph of $\widehat{G}_{adm,\lambda}$ block by block. The tables given below relate the output of such computations to nilpotent orbits, primitive ideals and Weyl group representations.

A.1. **Computational Scope.** The tables given in Appendix B are complete for the real reductive Lie groups G whose complexified Lie algebras coincide with one of the exceptional Lie algebras. In this section, I describe briefly how these tables were produced.

The first thing I did was to run the Atlas `blocksizes` command to identify all of the nontrivial blocks of representations for the each real forms of $G_{\mathbb{C}}$[1]. In the results appearing below, the real forms of $G_{\mathbb{C}}$ are listed in the left most column and corresponding dual real forms are listed in the first row. The integer entries in the interior correspond to the number of representations in the corresponding block.

- E_6 : (simply connected), split inner class.

	$\mathfrak{e}_6$	$E_6(\mathfrak{f}_4)$	$E_6(\mathbb{R})$
$E_6(F_4)$	0	0	45
$E_6(\mathbb{R})$	1	513	1881

- E_6 : (simply connected), compact inner class

	$E_6(F_4)$	$E_6(\mathbb{R})$
$\mathfrak{e}_6$	0	1
$E_6(\mathfrak{f}_4)$	0	513
$E_6(\mathbb{R})$	45	1881

---

[1]Parentheses are used to indicate the insignificant choices that were made in setting up the inner classes.

- E_7 : simply connected, (split inner class)

	$\mathfrak{e}_7$	$E_7\left(\mathfrak{e}_6.\mathfrak{u}\left(1\right)\right)$	$E_7\left(\mathfrak{so}\left(12\right).\mathfrak{su}\left(2\right)\right)$	$E_7\left(\mathbb{R}\right)$
$\mathfrak{e}_7$	0	0	0	1
$E_7\left(\mathfrak{e}_6.\mathfrak{u}\left(1\right)\right)$	0	0	315	3017
$E_7\left(\mathfrak{so}\left(12\right).\mathfrak{su}\left(2\right)\right)$	0	315	0	9576
$E_7\left(\mathbb{R}\right)$	1	1981	9576	24678

- E_7 : adjoint, (compact inner class)

	$\mathfrak{e}_7$	$E_7\left(\mathfrak{e}_6.\mathfrak{u}\left(1\right)\right)$	$E_7\left(\mathfrak{so}\left(12\right).\mathfrak{su}\left(2\right)\right)$	$E_7\left(\mathbb{R}\right)$
$\mathfrak{e}_7$	0	0	0	1
$E_7\left(\mathfrak{e}_6.\mathfrak{u}\left(1\right)\right)$	0	0	315	1981
$E_7\left(\mathfrak{so}\left(12\right).\mathfrak{su}\left(2\right)\right)$	0	315	0	9576
$E_7\left(\mathbb{R}\right)$	1	3017	9576	24678

- E_8 : (simply connected), (split inner class)

	$\mathfrak{e}_8$	$E_8\left(\mathfrak{e}_7.\mathfrak{su}\left(2\right)\right)$	$E_8\left(\mathbb{R}\right)$
$\mathfrak{e}_8$	0	0	1
$E_8\left(\mathfrak{e}_7.\mathfrak{su}\left(2\right)\right)$	0	3150	74410
$E_8\left(\mathbb{R}\right)$	1	73410	453060

- F_4 : (simply connected), (split inner class)

	$\mathfrak{f}_4$	$F_4\left(\mathfrak{so}\left(9\right)\right)$	$F_4\left(\mathbb{R}\right)$
$\mathfrak{f}_4$	0	0	1
$F_4\left(\mathfrak{so}\left(9\right)\right)$	0	0	15
$F_4\left(\mathbb{R}\right)$	1	15	336

- G_2 : (simply connected), (split inner class)

	$\mathfrak{g}_2$	$G_2\left(\mathbb{R}\right)$
$\mathfrak{g}_2$	0	1
$G_2\left(\mathbb{R}\right)$	1	12

A.2. **Cell Invariants.** On each of the blocks listed above, I then ran the Atlas `wcells` command. This command decomposes the vertices of a block into cells and then ascribes to each cell vertex its τ-invariant, edges, and edge-multiplicities.

With the `wcells` data in hand, I next proceeded to compute the following invariants, using Maple and John Stembridge's Coxeter/Weyl packages.

A.2.1. *Cardinality.*

Definition: The number of representations in a cell.

Algorithm: count

A.2.2. *Attached Nilpotent Orbit.*

Definition: The primitive ideals in $U\left(\mathfrak{g}\right)$ corresponding to the annihilators of the representations in a cell C have the share the same associated variety and that associated variety is the closure of a single nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}^*$. This orbit is the nilpotent orbit attached to a cell.

Algorithm: In the case of regular infinitesimal characters, the attached orbits are always special nilpotent orbits. These in turn can be uniquely specified by prescribing the minimal Richardson orbits whose closures

contain them and the Richardson orbits whose closures contain their Spaltenstein dual. This prescription in turn amounts to a specification of certain subsets of simple roots: for a Richardson orbit is an orbit of the form $ind_{l_\Gamma}^{\mathfrak{g}}(\mathbf{0})$, where Γ is a subset of the simple roots Π of $\mathfrak{g}$. The **tau signature** of a special orbit $\mathcal{O}$ is defined as the pair $(\tau(\mathcal{O}), \tau(\mathcal{O}^\vee))$ where

$$\tau(\mathcal{O}) = \left\{ \Gamma \subset \Pi \mid \mathcal{O} \subset \overline{ind_{l_\Gamma}^{\mathfrak{g}}(\mathbf{0})} \right\}$$

A similar τ-signature $(\tau(C), \tau^\vee(C))$ was defined for cells,

$$\tau(C) = \{ \Gamma \subset \Pi \mid \Gamma \text{ occurs in the set of } \tau\text{-invariants of } C \}$$
$$\tau^\vee(C) = \{ \Gamma \subset \Pi \mid \Gamma \text{ occurs in the set of complements of the } \tau\text{-invariants in } C \}$$

By results of Spaltenstein and David, a necessary and sufficient for a special nilpotent orbit $\mathcal{O}$ to be attached to a cell C is

$$(\tau(\mathcal{O}), \tau(\mathcal{O}^\vee)) = (\tau(C), \tau^\vee(C)) O$$

Orbits were attached to cells by explicitly computing the tau signatures of orbits and cells and then matching them up.

A.2.3. *Special Representations.*

Definition: The Springer correspondence attaches a unique representation of the Weyl group to each nilpotent orbit. The representations attached to special nilpotent orbits turn out to be special representations of the Weyl group. (I should point out that the notion of a special representation of W is independent of the Springer correspondence.)

Algorithm: Once nilpotent orbits were attached to cells, the tables in [Carter, *Finite Groups of Lie Type*] were used to assign special representations to cells.

A.2.4. τ_∞-*partitioning.*

Definition: Let $\tau_0(x)$ be the τ-invariant of a vertex x of a cell C, and let $\varepsilon(x)$ be the set of vertices y for which $x \to y$ is an edge. Set

$$\tau_1(x) = \{ \tau_0(y) \mid y \in \varepsilon(x) \}$$
$$\tau_2(x) = \{ \tau_1(y) \mid y \in \varepsilon(x) \}$$
$$\vdots$$
$$\tau_k(x) = \{ \tau_{k-1}(y) \mid y \in \varepsilon(x) \}$$

Let P_k be the partitioning of a cell corresponding to the equivalence relation

$$x \sim_K x' \iff \tau_i(x) = \tau_i(x') \quad \text{for } i = 0, 1, \ldots, k$$

Then P_k is a refinement of P_{k-1}. Since the cells are finite in cardinality, eventually $P_k = P_{k+1} = \cdots$ for all k sufficiently large. We call the resulting stable partitioning of C, the τ_∞ partitioning of C. It is compatible with the partitioning of a cell into subsets of representations sharing the same primitive ideal (i.e. the partitioning of C by the equivalence relation $x \sim_{PI} x' \iff Ann_{U(\mathfrak{g})}(x) = Ann_{U(\mathfrak{g})}(x')$).

Algorithm: straight-forward application of the definition to W-graph of a cell (from `wcells` data). The "answer" is most succinctly expressed as a partition of $|C|$; e.g. I use

$$\tau_\infty(C) = 1^{84} 2^{336}$$

to indicate a 756-element cell breaks up into 84 singlets and 336 doublets under the τ_∞-partitioning scheme.

Observation: In all cases the number of τ_∞-equivalence classes of in a cell coincides with the maximal number of distinct primitive ideals (= dimension of the special Weyl group representation attached to the nilpotent orbit attached to the cell) that can occur. Since the partitioning of the cell by primitive ideals is at worst a refinement of the τ_∞-partitioning, the fact that

$$\#\{\tau_\infty\text{-equivalence classes}\} = \text{maximal number of distinct primitive ideals that can occur in } C$$

implies that the τ_∞ partitioning and the partitioning by primitive ideals coincide.

A.2.5. *Cell Representation.*

Definition: The W-graph of a cell prescribes a representation of the Weyl group W of $\mathfrak{g}$, induced by the coherent continuation representation of W on the parameter set $\mathcal{HC}_\lambda$.

Algorithm: Let Γ_C be the free $\mathbb{Z}$-module freely generated by the irreducible x in C. For each $i \in \Pi$, define an action of the simple Weyl reflections on Γ_C by

$$T_i(x) = \begin{cases} -x & \text{if } i \in \tau(x) \\ x + \sum_{\substack{y \in \varepsilon(x) \\ i \notin \tau(y)}} m_{y \to x} y & \text{if } i \notin \tau(x) \end{cases}$$

This action on generators satisfies both quadratic relations and braid relations of W and so yeilds a representation of W. By explicitly constructing a set of $|C| \times |C|$ matrices implementing this action and then computing the traces of the representatives of each conjugacy class in W, the character of the cell representation can be computed. Expressing this character as a sum of irreducible characters yields the cell representation.

Observation: In writing down the cell representations, I utilized a parameterization of irreducible Weyl group representations by pairs (D, d) where D is the dimension of the representation and d is the degree at which the representation first occurs in $S(\mathfrak{h})$. (With just a couple of exceptions this data is sufficient to separate the representations of W.) This turned out to be a fortuitous choice for

- If σ_s is the special representation that occurs in a cell representation (each cell representation has one and only special representation as a constituent), then

$$\text{degree } d \text{ in which } \sigma_s \text{ first occurs in } S(\mathfrak{h}) = \left|\Delta^+\right| - \frac{1}{2}\dim(\mathcal{O}_C)$$

- If σ' is any other irreducible representation occuring in a cell representation then

$$\text{degree } d \text{ in which } \sigma' \text{ first occurs in } S(\mathfrak{h}) = \left|\Delta^+\right| - \frac{1}{2}\dim(\mathcal{O}_C) + \varepsilon$$

by which I mean σ' first occurs in $S(\mathfrak{h})$ at a degree that is only slightly higher than that of the special representation. In what follows, I shall make some wild speculations about what this might mean.

Appendix B. Tables

Below we give a series of tables for the various blocks of irreducible Harish-Chandra modules of the exceptional groups. Here is how the data is organized. Each row in each table corresponds to a particular cell in a particular block of an exceptional group. In cases where the adjoint group and the simply connected groups produce the same table we present only the table for the simply connected group. Similarly, whenever the same table results for the split and compact inner classes, we present only the table for the split inner class. The data in the columns of each table is as follows:

$\#$ the cell number as enumerated in the output of the atlas `extract-cells` command
$|C|$ the cardinality of the cell (i.e. the number of irreducible representations in the cell)

$\mathcal{O}_C$ the special orbit attached to the cell - computed by determining the minimal Richardson orbits containing the cell's orbit and the minimal Richardson orbits containing the orbit of the corresponding dual cell)

n the order in which the orbit $\mathcal{O}_C$ appears in the Collingwood and McGoverns tables of nilpotent orbits (pgs. $128 - 134$ of [CW]).

σ_{spec} the special representation attached to the cell - determined by applying the Springer correspondence to $\mathcal{O}_C$

$\#Prim\,(C)$ the number of primitive ideals in the cell (the number of distinct subcells in the τ_∞ partitioning of the cell)

$\mathcal{P}_{\tau_\infty}$ the τ_∞ partitioning of the cell written as an actual partition of $|C|$. E.g. $1^{20}2^{50}3^{10}$ means the τ_∞ partitioning of the cell decomposes it into 20 singlets, 50 doublets, and 10 triplets of representations sharing the same primitive ideal.

W-rep   the representation of the Weyl group carried by the cell. This is the coherent continuation representation inferred from the W-graph structure of the cell. It was computed by using the output of `extract-cells` to explicitly construct the (restriction of the) coherent continuation representation to the cell, computing the character of this representation by taking the traces of representatives of each conjugacy class in W, and then writing this character into a sum irreducible characters. I use the notation of Carter (*Finite Groups of Lie Type*, John Wiley and Sons, New York, (1985)) to specify Weyl group representations. In this notation, $\phi_{D,d}$ denotes the Weyl group representation of dimension D that first appears in $S(\mathfrak{h}^*)$ (as homogeneous harmonic polynomials) of degree d; primes "$\prime$" are used to distinguish inequivalent representations with the same D and d.

The last series of table contains the analogous results for the complex exceptional groups, excluding $E_8\,(\mathbb{C})$.

B.1. G_2.

11 block : $G\,(\mathbb{R})\,, G\,(\mathbb{R})$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	G_2	5	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	5	$G_2(a_1)$	4	$\phi_{2,1}$	2	$2^1 3^1$	$\phi_{2,1} + \phi_{2,2} + \phi'_{1,3}$
2	5	$G_2(a_1)$	4	$\phi_{2,1}$	2	$2^1 3^1$	$\phi_{2,1} + \phi_{2,2} + \phi''_{1,3}$
3	1	0	1	$\phi_{1,6}$	1	1^1	$\phi_{1,6}$

B.2. F_4.

12 block : $F_4\,(\mathfrak{so}(9))\,, F_4\,(\mathbb{R})$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	8	$\tilde{A}_2$	6	$\phi'_{8,9}$	8	1^8	$\phi'_{8,9}$
1	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13} + \phi'_{2,16}$
2	1	0	1	$\phi_{1,24}$	1	1^1	$\phi_{1,24}$

21 block : $F_4\,(\mathbb{R})\,, F_4\,(\mathfrak{so}(9))$ block

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1} + \phi'_{2,4}$
2	8	C_3	13	$\phi'_{8,3}$	8	1^8	$\phi'_{8,3}$

22 block : $F_4(\mathbb{R}), F_4(\mathbb{R})$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1}+\phi''_{2,4}$
3	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1}+\phi''_{2,4}$
4	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
5	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
11	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
12	8	C_3	13	$\phi'_{8,3}$	8	1^8	$\phi'_{8,3}$
2	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
6	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
7	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
8	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
9	57	$F_4(a_3)$	11	$\phi_{12,4}$	12	$2^3 3^1 5^3 6^4 9^1$	$\phi_{12,4}+\phi_{16,5}+2\phi''_{9,6}+\phi''_{6,6}+\phi''_{4,7}+\phi''_{1,12}$
13	47	$F_4(a_3)$	11	$\phi_{12,4}$	12	$5^4 4^4 3^3 2^1$	$\phi_{12,4}+\phi_{16,5}+\phi''_{9,6}+\phi'_{6,6}+\phi''_{4,7}$
14	72	$F_4(a_3)$	11	$\phi_{12,4}$	12	$4^6 6^2 9^4$	$\phi_{12,4}+2\phi_{16,5}+\phi'_{9,6}+\phi''_{9,6}+\phi''_{6,6}+\phi_{4,8}$
18	8	$\tilde{A}_2$	6	$\phi'_{8,9}$	8	1^8	$\phi'_{8,9}$
10	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
15	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
16	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
21	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
17	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
19	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
20	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
22	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13}+\phi''_{2,16}$
23	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13}+\phi''_{2,16}$
24	1	0	1	$\phi_{1,24}$	1	1^1	$\phi_{1,24}$

B.3. E_6 : compact inner class.

11 block : $E_6(\mathfrak{f}_4), E_6(\mathfrak{f}_4)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
2	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
3	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
1	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
4	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
6	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
8	45	A_2	5	$\phi_{30,15}$	30	$1^{15} 2^{15}$	$\phi_{30,15}+\phi_{15,17}$
5	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
7	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
9	20	$2A_1$	3	$\phi_{20,10}$	20	1^{20}	$\phi_{20,20}$
10	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
11	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
12	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

20 block : $E_6(\mathbb{R}), \mathfrak{e}_6$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
2	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$

21 block : $E_6\,(\mathbb{R})\,, E_6\,(\mathfrak{f}_4)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
3	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
1	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
5	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,4}$
11	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,4}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
7	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
12	60	$A_4 + A_1$	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
13	60	$A_4 + A_1$	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
2	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
9	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
8	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
14	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
10	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7} + \phi_{90,8} + \phi_{60,8}$
15	150	$D_4(a_1)$	12	$\phi_{80,7}$	80	$1^{20}2^{50}3^{10}$	$\phi_{80,7} + \phi_{60,8} + \phi_{10,9}$
16	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7} + \phi_{90,8} + \phi_{60,8}$
19	81	$A_3 + A_1$	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
21	81	$A_3 + A_1$	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
17	60	$A_2 + 2A_1$	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
18	60	$A_2 + 2A_1$	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
22	60	$A_2 + 2A_1$	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
23	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
24	64	$A_2 + A_1$	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
25	64	$A_2 + A_1$	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
20	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15} + \phi_{15,16}$
26	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15} + \phi_{15,16}$
27	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15} + \phi_{15,16}$
28	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
29	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
30	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
31	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

B.4. E_6 : split inner class.

02 block : $\mathfrak{e}_6, E\,(\mathbb{R})$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
1	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
2	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

11 block : $E_6\left(\mathfrak{f}_4\right), E_6\left(\mathfrak{f}_4\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
2	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
3	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
5	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
7	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,5}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
8	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
9	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
10	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
11	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
12	81	A_3	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,6}$

12 block : $E_6\left(\mathfrak{f}_4\right), E_6\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
2	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
3	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,4}$
5	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,4}$
15	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3} + \phi_{15,4}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
7	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
9	60	$A_4 + A_1$	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
10	60	$A_4 + A_1$	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
11	60	$A_4 + A_1$	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
8	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
12	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
16	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
13	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7} + \phi_{90,8} + \phi_{60,8}$
14	150	$D_4(a_1)$	12	$\phi_{80,7}$	80	$1^{20}2^{50}3^{10}$	$\phi_{80,7} + \phi_{60,8} + \phi_{10,9}$
17	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7} + \phi_{90,8} + \phi_{60,8}$
18	81	$A_3 + A_1$	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
23	81	$A_3 + A_1$	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
19	60	$A_2 + 2A_1$	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
20	60	$A_2 + 2A_1$	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
22	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
27	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
24	64	$A_2 + A_1$	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
25	64	$A_2 + A_1$	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
21	45	A_2	5	$\phi_{30,15}$	60	$1^{15}2^{15}$	$\phi_{30,15.} + \phi_{15,16}$
26	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15.} + \phi_{15,16}$
28	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
29	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
30	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
31	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

B.5. E_7 : **simply connected.**

12 block : $E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right), E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right)$

Block	#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
12	0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
12	1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
12	2	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

13 block : $E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
2	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
3	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
4	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
5	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
6	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
11	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
9	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
10	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
12	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
15	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	77	A_2	6	$\phi_{56,30}$	56	$1^{35}2^{21}$	$\phi_{56,30}+\phi_{21,33}$
7	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
8	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
13	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
16	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
14	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
17	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
19	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
20	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
21	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
22	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

21 block : $E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right), E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
2	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

23 block : $E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
2	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
4	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
5	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
11	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
12	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
6	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
13	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
7	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
14	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
15	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
3	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
8	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
9	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
19	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
20	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
16	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
22	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
17	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
21	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
23	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
24	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
25	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
26	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
27	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
28	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
29	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
30	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
31	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

31 block : $E_7(\mathbb{R})$, $E_7(\mathfrak{e}_6 + \mathfrak{u}(1))$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
5	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
7	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
6	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
14	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
8	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4} + \phi_{105,5}$
11	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
9	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
10	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
12	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
15	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
13	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
16	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

32 block : $E_7(\mathbb{R})$, $E_7(\mathfrak{so}(12) + \mathfrak{su}(2))$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
13	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
6	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4} + \phi_{105,5}$
7	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
10	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
14	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
8	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
11	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7} + \phi_{280,8} + \phi_{280,9}$
12	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7} + \phi_{280,8} + \phi_{70,9}$
21	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7} + \phi_{280,8} + \phi_{280,9}$
16	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
17	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
23	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
18	1024	$A_4 + A_1$	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11} + \phi_{512,12}$
22	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
29	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
19	210	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
24	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13} + \phi_{336,14}$
20	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
25	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
26	189	$(A_3 + A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
30	189	$(A_3 + A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
27	105	$A_2 + 3A_1$	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
28	135	$A_2 + A_1$	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25} + \phi_{15,28}$
31	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

33 block : $E_7\left(\mathbb{R}\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
3	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
6	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
7	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
10	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
11	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3} + \phi_{35,4}$
8	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
9	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4} + \phi_{105,5}$
12	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4} + \phi_{15.7}$
13	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
14	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
15	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
18	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
22	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
16	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
24	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
17	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
23	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
19	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
27	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
20	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7} + \phi_{280,8} + \phi_{280,9}$
25	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7} + \phi_{280,8} + \phi_{280,9}$
29	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7} + \phi_{280,8} + \phi_{70,9}$
30	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7} + \phi_{280,8} + \phi_{70,9}$
26	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
28	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8} + \phi_{216,9}$
31	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8} + \phi_{216,9}$
21	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
32	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
33	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
35	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
42	378	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
36	210	$A_4 + A_2$	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
37	210	$A_4 + A_2$	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
34	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
39	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
44	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10} + \phi_{84,12}$
38	1024	$A_4 + A_1$	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11} + \phi_{512,12}$
40	1024	$A_4 + A_1$	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11} + \phi_{512,12}$
45	1024	$A_4 + A_1$	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11} + \phi_{512,12}$

33 block : $E_7\left(\mathbb{R}\right), E_7\left(\mathbb{R}\right)$ cont'd

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
52	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
53	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
41	210	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
43	210	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
46	210	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
49	210	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
54	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13} + \phi_{84,15}$
57	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13} + \phi_{336,14}$
55	378	$A_3 + A_2$	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
56	378	$A_3 + A_2$	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
58	378	$A_3 + A_2$	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
63	378	$A_3 + A_2$	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
47	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
50	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
59	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
60	621	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15} + \phi_{216,16}$
61	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
62	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16} + \phi_{280,17} + \phi_{70,18}$
64	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16} + \phi_{280,17} + \phi_{280,18}$
69	189	$(A_3 + A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
70	189	$(A_3 + A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
48	105	$A_2 + 3A_1$	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
51	105	$A_2 + 3A_1$	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
65	105	$A_2 + 3A_1$	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
67	105	$A_2 + 3A_1$	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
71	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
72	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
66	189	$A_2 + 2A_1$	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
68	189	$A_2 + 2A_1$	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
73	189	$A_2 + 2A_1$	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
74	189	$A_2 + 2A_1$	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
75	135	$A_2 + A_1$	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25} + \phi_{15,28}$
76	135	$A_2 + A_1$	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25} + \phi_{15,28}$
78	135	$A_2 + A_1$	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25} + \phi_{15,28}$
77	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30} + \phi_{35,31}$
79	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30} + \phi_{35,31}$
80	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
81	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
82	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
83	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
84	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

B.6. E_7 : **adjoint.**

12 block : $E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right), E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
2	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

13 block : $E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
1	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
2	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
3	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
6	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
7	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
5	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
8	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
11	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
4	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
9	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
10	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
12	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
13	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
14	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
15	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$
16	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

21 block : $E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right), E_7\left(\mathfrak{e}_6+\mathfrak{u}(1)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
2	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

23 block : $E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	$\sigma_{spec}\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	$\mathcal{P}^t_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
2	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
4	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
5	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
11	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
12	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
6	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
13	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
7	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
14	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
15	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
3	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
8	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
9	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
19	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
20	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
16	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
22	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
17	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
21	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
23	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
24	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
25	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
26	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
27	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
28	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
29	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
30	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
31	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

31 block : $E_7\left(\mathbb{R}\right), E_7\left(\mathfrak{e}_6 + \mathfrak{u}(1)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
6	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
8	77	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{35}2^{21}$	$\phi_{56,3} + \phi_{21,6}$
5	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
7	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
17	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
18	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
9	225	$E_6\left(a_1\right)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4} + \phi_{105,5}$
10	225	$E_6\left(a_1\right)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4} + \phi_{105,5}$
12	210	$D_6\left(a_1\right)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
13	210	$D_6\left(a_1\right)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
11	168	$D_5 + A_1$	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
14	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
15	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
19	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
20	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
16	756	$D_5\left(a_1\right)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10} + \phi_{336,11}$
21	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
22	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

32 block : $E_7\left(\mathbb{R}\right), E_7\left(\mathfrak{so}(12)+\mathfrak{su}(2)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7\left(a_1\right)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
13	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
6	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
7	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
10	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
14	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
8	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
11	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
12	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
21	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
16	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
17	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
23	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
18	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
22	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
29	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
19	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
24	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
20	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
25	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
26	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
30	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
27	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
28	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
31	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

33 block : $E_7\left(\mathbb{R}\right), E_7\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7\left(a_1\right)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
5	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
7	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
6	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
8	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
11	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
14	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
16	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
12	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
13	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
17	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
21	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
24	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
18	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
26	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
20	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
23	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
22	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
25	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
28	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
19	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
29	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
30	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
31	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
33	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
34	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
35	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
36	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
32	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
38	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
37	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
39	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
40	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$

33 block : $E_7(\mathbb{R}), E_7(\mathbb{R})$ cont'd

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
45	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
41	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
44	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
46	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
47	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
49	504	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{84,15}$
48	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
50	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
51	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
57	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
59	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
42	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
52	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
53	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
55	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
54	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
56	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
58	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
60	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
63	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
43	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
61	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
64	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
65	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
66	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
69	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
62	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
67	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
68	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
70	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
71	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
72	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
75	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
73	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
74	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
76	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
77	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
78	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
79	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
80	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
81	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
82	7	A_1	2	$\phi_{7,46}$	7	1^7	$\phi_{7,46}$
83	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$
84	1	0	1	$\phi_{1,63}$	1	1^1	$\phi_{1,63}$

B.7. E_8.

11 block : $E_8\left(\mathfrak{e}_7+\mathfrak{su}(2)\right), E_8\left(\mathfrak{e}_7+\mathfrak{su}(2)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
1	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
2	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$

12 block : $E_8\left(\mathfrak{e}_7+\mathfrak{su}(2)\right), E_8\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
1	3500	$E_6\left(a_1\right)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
5	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
2	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
3	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
4	8800	$E_6\left(a_3\right)$	33	$\phi_{5600,21}$	5600	$1^{2400}3^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
6	8800	$E_6\left(a_3\right)$	33	$\phi_{5600,21}$	5600	$1^{2400}3^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
7	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
8	4900	$D_5\left(a_1\right)$	26	$\phi_{2800,25}$	2800	$1^{700}2^{2100}$	$\phi_{2800,25}+\phi_{2100,28}$
9	8192	A_4+A_1	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26}+\phi_{4096,27}$
11	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
14	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
18	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
10	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
13	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
12	3192	$D_4\left(a_1\right)$	14	$\phi_{1400,37}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{448,39}$
15	3752	$D_4\left(a_1\right)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{1008,39}$
19	3752	$D_4\left(a_1\right)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{1008,39}$
20	1000	$2A_2$	11	$\phi_{700,42}$	700	$1^{400}2^{300}$	$\phi_{700,42}+\phi_{300,44}$
16	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
21	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
22	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
24	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
25	370	A_2+A_1	7	$\phi_{210,52}$	210	$1^{50}2^{160}$	$\phi_{210,52}+\phi_{160,55}$
17	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
23	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
26	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
27	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
28	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
29	8	A_1	2	$\phi_{8,91}$	8	1^8	$\phi_{8,91}$
30	1	1	1	$\phi_{1,120}$	1	1^1	$\phi_{1,120}$

21 block : $E_8\left(\mathbb{R}\right), E_8\left(\mathfrak{e}_7+\mathfrak{su}(2)\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_8	70	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	8	$E_8\left(a_1\right)$	69	$\phi_{8,1}$	8	1^8	$\phi_{8,1}$
2	35	$E_8\left(a_2\right)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
3	35	$E_8\left(a_2\right)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
4	196	$E_8\left(a_3\right)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
5	196	$E_8\left(a_3\right)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
13	196	$E_8\left(a_3\right)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
6	370	$E_8\left(a_4\right)$	66	$\phi_{210,4}$	210	$1^{50}2^{160}$	$\phi_{210,4}+\phi_{160,7}$
7	560	$E_8\left(b_4\right)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
9	560	$E_8\left(b_4\right)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
8	1000	$E_8\left(a_5\right)$	63	$\phi_{700,6}$	700	$1^{400}2^{300}$	$\phi_{700,6}+\phi_{300,8}$
10	567	$E_7\left(a_1\right)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
14	567	$E_7\left(a_1\right)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
11	3752	$E_8\left(b_5\right)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
12	3192	$E_8\left(b_5\right)$	61	$\phi_{1400,7}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{448,9}$
20	3752	$E_8\left(b_5\right)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
15	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
16	3240	$E_7\left(a_3\right)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
22	3240	$E_7\left(a_3\right)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
17	8192	$E_6\left(a_1\right)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
21	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
28	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
18	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
23	4900	$E_6\left(a_1\right)$	47	$\phi_{2800,13}$	2800	$1^{700}2^{2100}$	$\phi_{2800,13}+\phi_{2100,16}$
19	8800	$D_6\left(a_1\right)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
24	8800	$D_6\left(a_1\right)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
25	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
29	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
26	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
27	3500	$D_5\left(a_1\right)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25}+\phi_{700,28}$
30	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$

22 block : $E_8\left(\mathbb{R}\right), E_8\left(\mathbb{R}\right)$

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\left(C\right)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_8	70	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	8	$E_8\left(a_1\right)$	69	$\phi_{8,1}$	8	1^8	$\phi_{8,1}$
2	35	$E_8\left(a_2\right)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
3	196	$E_8\left(a_3\right)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84.4}$
5	196	$E_8\left(a_3\right)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84.4}$
4	260	$E_8\left(a_4\right)$	66	$\phi_{210,4}$	210	$1^{160}2^{50}$	$\phi_{210,4}+\phi_{50,8}$
7	260	$E_8\left(a_4\right)$	66	$\phi_{210,4}$	210	$1^{160}2^{50}$	$\phi_{210,4}+\phi_{50,8}$
6	560	$E_8\left(b_4\right)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
8	560	$E_8\left(b_4\right)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
9	560	$E_8\left(b_4\right)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
10	1100	$E_8\left(a_5\right)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
13	1100	$E_8\left(a_5\right)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
17	1100	$E_8\left(a_5\right)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
11	567	$E_7\left(a_1\right)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
12	3192	$E_8\left(b_5\right)$	61	$\phi_{1400,7}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{448,9}$
14	3752	$E_8\left(b_5\right)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
15	4025	$E_8\left(a_6\right)$	59	$\phi_{1400,8}$	1400	$2^{175}3^{1225}$	$\phi_{1400,8}+\phi_{1575,10}+\phi_{1050,10}$
18	2625	$E_8\left(a_6\right)$	59	$\phi_{1400,8}$	1400	$1^{350}2^{875}3^{175}$	$\phi_{1400,8}+\phi_{1050,10}+\phi_{175,12}$
16	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
19	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
20	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
21	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
22	3240	$D_7\left(a_1\right)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
23	3640	$E_8\left(b_6\right)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
25	3640	$E_8\left(b_6\right)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
27	3640	$E_8\left(b_6\right)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
24	3240	$E_7\left(a_3\right)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
29	3240	$E_7\left(a_3\right)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
26	8192	$E_6\left(a_1\right)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
30	8192	$E_6\left(a_1\right)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
28	7560	$D_7\left(a_2\right)$	51	$\phi_{4200,12}$	4200	$1^{840}2^{3360}$	$\phi_{4200,12}+\phi_{3360,13}$
31	5040	$D_7\left(a_2\right)$	51	$\phi_{4200,12}$	4200	$1^{3360}2^{840}$	$\phi_{4200,12}+\phi_{840,14}$
35	7560	$D_7\left(a_2\right)$	51	$\phi_{4200,12}$	4200	$1^{840}2^{3360}$	$\phi_{4200,12}+\phi_{3360,13}$
33	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
32	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
38	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
39	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
45	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
34	3500	$E_6\left(a_1\right)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
37	3500	$E_6\left(a_1\right)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
36	6075	$E_7\left(a_4\right)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
41	6075	$E_7\left(a_4\right)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
43	6075	$E_7\left(a_4\right)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
40	2835	A_6+A_1	45	$\phi_{2835,14}$	2835	1^{2835}	$\phi_{2835,14}$
44	8800	$D_6\left(a_1\right)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
49	8800	$D_6\left(a_1\right)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
42	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
46	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
48	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
50	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
47	22778	$E_8\left(a_7\right)$	42	$\phi_{4480,16}$	4480	$2^{70}3^{378}4^{1092}$ $\cdot5^{924}6^{1596}7^{420}$	$\phi_{4480,16}+\phi_{7168,17}+\phi_{3150,18}$ $+\phi_{4200,18}+\phi_{1344,19}+\phi_{2016,19}$ $+\phi_{420,20}$
51	38766	$E_8\left(a_7\right)$	42	$\phi_{4480,16}$	4480	$5^{420}7^{1134}$ $\cdot8^{1596}12^{1330}$	$\phi_{4480,16}+\phi_{7168,17}+2\phi_{4200,18}$ $+\phi_{4536,18}+\phi_{5670,18}+\phi_{1344,19}$ $+\phi_{5600,19}+\phi_{1400,20}+\phi_{168,24}$
53	46676	$E_8\left(a_7\right)$	42	$\phi_{4480,16}$	4480	$6^{420}7^{756}8^{168}$ $\cdot9^{378}11^{70}$ $\cdot12^{1596}13^{1092}$	$\phi_{4480,16}+2\phi_{7168,17}+\phi_{3150,18}$ $+\phi_{4200,18}+\phi_{4536,18}+\phi_{5670,18}$ $+\phi_{2016,19}+\phi_{5600,19}+\phi_{2688,20}$

22 block : $E_8(\mathbb{R}), E_8(\mathbb{R})$, cont'd

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
54	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
55	8800	$E_6(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21} + \phi_{3200,22}$
63	8800	$E_6(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21} + \phi_{3200,22}$
52	4200	$D_4 + A_2$	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
57	4200	$D_4 + A_2$	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
59	4200	$D_4 + A_2$	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
62	4200	$D_4 + A_2$	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
60	2835	$A_4 + A_2 + A_1$	31	$\phi_{2835,22}$	2835	1^{2835}	$\phi_{2835,22}$
58	6075	$D_5(a_1) + A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
64	6075	$D_5(a_1) + A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
65	6075	$D_5(a_1) + A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
56	4536	$A_4 + A_2$	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
66	4536	$A_4 + A_2$	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
68	4536	$A_4 + A_2$	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
69	4536	$A_4 + A_2$	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
61	7560	$A_4 + 2A_1$	27	$\phi_{4200,24}$	4200	$1^{840}2^{3360}$	$\phi_{4200,24} + \phi_{3360,25}$
67	7560	$A_4 + 2A_1$	27	$\phi_{4200,24}$	4200	$1^{840}2^{3360}$	$\phi_{4200,24} + \phi_{3360,25}$
70	5040	$A_4 + 2A_1$	27	$\phi_{4200,24}$	4200	$1^{3360}2^{840}$	$\phi_{4200,24} + \phi_{840,26}$
71	3500	$D_5(a_1)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25} + \phi_{700,28}$
79	3500	$D_5(a_1)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25} + \phi_{700,28}$
72	8192	$A_4 + A_1$	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26} + \phi_{4096,27}$
80	8192	$A_4 + A_1$	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26} + \phi_{4096,27}$
73	3640	$D_4(a_1) + A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28} + \phi_{1400,29}$
78	3640	$D_4(a_1) + A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28} + \phi_{1400,29}$
81	3640	$D_4(a_1) + A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28} + \phi_{1400,29}$
74	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30} + \phi_{972,32}$
82	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30} + \phi_{972,32}$
75	3240	$A_3 + A_2$	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
76	3240	$A_3 + A_2$	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
83	3240	$A_3 + A_2$	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
84	3240	$A_3 + A_2$	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
86	3240	$A_3 + A_2$	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
87	4025	$D_4(a_1) + A_1$	18	$\phi_{1400,32}$	1400	$2^{175}3^{1225}$	$\phi_{1400,32} + \phi_{1575,34} + \phi_{1050,34}$
89	2625	$D_4(a_1) + A_1$	18	$\phi_{1400,32}$	1400	$1^{350}2^{875}3^{175}$	$\phi_{1400,32} + \phi_{1050,34} + \phi_{175,36}$
85	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
88	3752	$D_4(a_1)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37} + \phi_{1344,38} + \phi_{1008,39}$
90	3192	$D_4(a_1)$	14	$\phi_{1400,37}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,37} + \phi_{1344,38} + \phi_{448,39}$
77	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42} + \phi_{400,43}$
91	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42} + \phi_{400,43}$
92	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42} + \phi_{400,43}$
94	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
93	560	$A_2 + 2A_1$	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
95	560	$A_2 + 2A_1$	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
96	560	$A_2 + 2A_1$	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
97	260	$A_2 + A_1$	7	$\phi_{210,52}$	210	$1^{160}2^{50}$	$\phi_{210,52} + \phi_{50,56}$
99	260	$A_2 + A_1$	7	$\phi_{210,52}$	210	$1^{160}2^{50}$	$\phi_{210,52} + \phi_{50,56}$
98	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63} + \phi_{84,64}$
100	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63} + \phi_{84,64}$
101	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
102	8	A_1	2	$\phi_{8,91}$	8	1^8	$\phi_{8,91}$
103	1	0	1	$\phi_{1,120}$	1	1^1	$\phi_{1,120}$

B.8. $G_2(\mathbb{C})$.

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	G_2	5	$\phi_{1,0}$	1	1^1	$\left(\phi_{1,0}\right)^2$
2	10	$G_2(a_1)$	4	$\phi_{2,1}$	4	$2^2 3^2$	$\left(\phi_{2,1}\right)^2 + \left(\phi_{2,2}\right)^2 + \left(\phi'_{1,3}\right)^2 + \left(\phi''_{1,3}\right)^2$
3	1	0	1	$\phi_{1,6}$	1	1^1	$\left(\phi_{1,6}\right)^2$

B.9. $F_4(\mathbb{C})$.

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$\left(\phi_{1,0}\right)^2$
1	24	$F_4(a_1)$	15	$\phi_{4,1}$	16	$1^8 2^8$	$\left(\phi_{4,1}\right)^2 + \left(\phi'_{2,4}\right)^2 + \left(\phi''_{2,4}\right)^2$
2	81	$F_4(a_2)$	14	$\phi_{9,2}$	81	1^{81}	$\left(\phi_{9,2}\right)^2$
3	64	B_3	12	$\phi''_{8,3}$	64	1^{64}	$\left(\phi''_{8,3}\right)^2$
4	64	C_3	13	$\phi'_{8,3}$	64	1^{64}	$\left(\phi'_{8,3}\right)^2$
5	684	$F_4(a_3)$	11	$\phi_{12,4}$	144	$2^{12}3^{20}4^{48}5^{30}6^{16}9^{18}$	$\left(\phi_{12,4}\right)^2 + \left(\phi_{16,5}\right)^2 + \left(\phi'_{6,6}\right)^2$ $+ \left(\phi''_{6,6}\right)^2 + \left(\phi'_{9,6}\right)^2 + \left(\phi''_{9,6}\right)^2$ $+ \left(\phi'_{4,7}\right)^2 + \left(\phi''_{4,7}\right)^2 + \left(\phi_{4,8}\right)^2$ $+ \left(\phi'_{1,12}\right)^2 + \left(\phi''_{1,12}\right)^2$
6	64	A_2	5	$\phi''_{8,9}$	64	1^{64}	$\left(\phi''_{8,9}\right)^2$
7	64	$\widetilde{A}_2$	6	$\phi'_{8,9}$	64	1^{64}	$\left(\phi'_{8,9}\right)^2$
8	81	$A_1 + \tilde{A}_1$	4	$\phi_{9,10}$	81	1^{81}	$\left(\phi_{9,10}\right)^2$
9	24	$\tilde{A}_1$	3	$\phi_{4,13}$	16	$1^8 2^8$	$\left(\phi_{4,13}\right)^2 + \left(\phi'_{2,16}\right)^2 + \left(\phi''_{2,16}\right)^2$
10	1	0	1	$\phi_{1,24}$	1	1^1	$\left(\phi_{1,24}\right)^2$

B.10. $E_6(\mathbb{C})$.

#	$|C|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\left(\phi_{1,0}\right)^2$
1	36	$E_6(a_1)$	20	$\phi_{6,1}$	36	1^{36}	$\left(\phi_{6,1}\right)^2$
2	400	D_5	19	$\phi_{20,2}$	400	1^{400}	$\left(\phi_{20,2}\right)^2$
3	1350	$E_6(a_3)$	18	$\phi_{30,3}$	900	$1^{450}2^{450}$	$\left(\phi_{30,3}\right)^2 + \left(\phi_{15,4}\right)^2 + \left(\phi_{15,5}\right)^2$
4	4096	$D_5(a_1)$	17	$\phi_{64,4}$	4096	1^{4096}	$\left(\phi_{64,4}\right)^2$
5	3600	$A_4 + A_1$	15	$\phi_{60,5}$	3600	1^{3600}	$\left(\phi_{60,5}\right)^2$
6	6561	A_4	13	$\phi_{81,6}$	6561	1^{6561}	$\left(\phi_{81,6}\right)^2$
7	576	D_4	13	$\phi_{24,6}$	576	1^{576}	$\left(\phi_{24,6}\right)^2$
8	18600	$D_4(a_1)$	12	$\phi_{80,7}$	6400	$1^{400}2^{1000}3^{4600}6^{400}$	$\left(\phi_{80,7}\right)^2 + \left(\phi_{90,8}\right)^2 + \left(\phi_{60,8}\right)^2$ $+ \left(\phi_{10,9}\right)^2 + \left(\phi_{20,10}\right)^2$
9	6561	$A_3 + A_1$	9	$\phi_{81,10}$	6561	1^{6561}	$\left(\phi_{81,10}\right)^2$
10	3600	$A_2 + 2A_1$	8	$\phi_{60,11}$	3600	1^{3600}	$\left(\phi_{60,11}\right)^2$
11	576	$2A_2$	7	$\phi_{24,12}$	576	1^{576}	$\left(\phi_{24,12}\right)^2$
12	4096	$A_2 + A_1$	6	$\phi_{64,13}$	4096	1^{4096}	$\left(\phi_{64,13}\right)^2$
13	1350	A_2	5	$\phi_{30,15}$	900	$1^{450}2^{450}$	$\left(\phi_{30,15}\right)^2 + \left(\phi_{15,16}\right)^2 + \left(\phi_{15,17}\right)^2$
14	400	$2A_1$	3	$\phi_{20,20}$	400	1^{400}	$\left(\phi_{20,20}\right)^2$
15	36	A_1	2	$\phi_{6,25}$	36	1^{36}	$\left(\phi_{6,25}\right)^2$
16	1	0	1	$\phi_{1,36}$	1	1^1	$\left(\phi_{1,36}\right)^2$

B.11. $E_7(\mathbb{C})$.

#	$\lvert C \rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\left(\phi_{1,0}\right)^2$
1	49	$E_7(a_1)$	44	$\phi_{7,1}$	49	1^{49}	$\left(\phi_{7,1}\right)^2$
2	729	$E_7(a_2)$	43	$\phi_{27,2}$	729	1^{729}	$\left(\phi_{27,2}\right)^2$
3	4802	$E_7(a_3)$	42	$\phi_{56,3}$	3136	$1^{1470}2^{1666}$	$\left(\phi_{56,3}\right)^2 + \left(\phi_{35,4}\right)^2 + \left(\phi_{21,6}\right)^2$
4	441	E_6	41	$\phi_{21,3}$	441	1^{441}	$\left(\phi_{21,3}\right)^2$
5	25650	$E_6(a_1)$	40	$\phi_{120,4}$	14400	$1^{3150}2^{11250}$	$\left(\phi_{120,4}\right)^2 + \left(\phi_{105,5}\right)^2 + \left(\phi_{15,7}\right)^2$
6	35721	$E_7(a_4)$	38	$\phi_{189,5}$	35721	1^{35721}	$\left(\phi_{189,5}\right)^2$
7	44100	$D_6(a_1)$	37	$\phi_{210,6}$	44100	1^{44100}	$\left(\phi_{210,6}\right)^2$
8	28224	$D_5 + A_1$	36	$\phi_{168,6}$	28224	1^{28224}	$\left(\phi_{168,6}\right)^2$
9	11025	A_6	35	$\phi_{105,6}$	11025	1^{11025}	$\left(\phi_{105,6}\right)^2$
10	262150	$E_7(a_5)$	34	$\phi_{315,7}$	99225	$1^{4900}2^{29400}3^{63700}6^{1225}$	$\left(\phi_{315,7}\right)^2 + \left(\phi_{280,8}\right)^2 + \left(\phi_{280,9}\right)^2 + \left(\phi_{35,13}\right)^2$
11	35721	D_5	33	$\phi_{189,7}$	35721	1^{35721}	$\left(\phi_{189,7}\right)^2$
12	246402	$E_6(a_3)$	32	$\phi_{405,8}$	164025	$1^{81648}2^{82377}$	$\left(\phi_{405,8}\right)^2 + \left(\phi_{216,9}\right)^2 + \left(\phi_{189,10}\right)^2$
13	142884	$D_5(a_1) + A_1$	30	$\phi_{378,9}$	142884	1^{142884}	$\left(\phi_{378,9}\right)^2$
14	296352	$D_5(a_1)$	26	$\phi_{420,10}$	176400	$1^{56448}2^{119952}$	$\left(\phi_{210,10}\right)^2 + \left(\phi_{336,11}\right)^2 + \left(\phi_{84,12}\right)^2$
15	44100	$A_4 + A_2$	26	$\phi_{210,10}$	44100	1^{44100}	$\left(\phi_{210,10}\right)^2$
16	524288	$A_4 + A_1$	25	$\phi_{512,11}$	262144	1^{262144}	$\left(\phi_{512,11}\right)^2$
17	11025	$(A_5)''$	23	$\phi_{105,12}$	11025	1^{11025}	$\left(\phi_{105,12}\right)^2$
18	296352	A_4	21	$\phi_{420,13}$	176400	$1^{56448}2^{119952}$	$\left(\phi_{420,13}\right)^2 + \left(\phi_{336,14}\right)^2 + \left(\phi_{84,15}\right)^2$
19	44100	$A_3 + A_2 + A_1$	22	$\phi_{210,13}$	44100	1^{44100}	$\left(\phi_{210,13}\right)^2$
20	142884	$A_3 + A_2$	20	$\phi_{378,14}$	142884	1^{142884}	$\left(\phi_{378,14}\right)^2$
21	246402	$D_4(a_1) + A_1$	19	$\phi_{405,15}$	164025	$1^{81648}2^{82377}$	$\left(\phi_{405,15}\right)^2 + \left(\phi_{216,16}\right)^2 + \left(\phi_{189,17}\right)^2$
22	11025	D_4	18	$\phi_{105,15}$	11025	1^{11025}	$\left(\phi_{105,15}\right)^2$
23	262150	$D_4(a_1)$	16	$\phi_{315,16}$	99225	$1^{4900}2^{29400}3^{63700}6^{1225}$	$\left(\phi_{315,16}\right)^2 + \left(\phi_{280,17}\right)^2 + \left(\phi_{280,18}\right)^2 + \left(\phi_{70,,18}\right)^2 + \left(\phi_{35,22}\right)^2$
24	35721	$(A_3 + A_1)''$	13	$\phi_{189,20}$	35721	1^{35721}	$\left(\phi_{189,20}\right)^2$
25	11025	$A_2 + 3A_1$	12	$\phi_{105,21}$	11025	1^{11025}	$\left(\phi_{105,21}\right)^2$
26	28224	$2A_2$	11	$\phi_{168,21}$	28224	1^{28224}	$\left(\phi_{168,21}\right)^2$
27	44199	A_3	10	$\phi_{210,21}$	44100	1^{44100}	$\left(\phi_{210,21}\right)^2$
28	35721	$A_2 + 2A_1$	9	$\phi_{189,22}$	35721	1^{35721}	$\left(\phi_{189,22}\right)^2$
29	25650	$A_2 + A_1$	8	$\phi_{120,25}$	14400	1^{14400}	$\left(\phi_{120,25}\right)^2$
30	4802	A_2	6	$\phi_{56,30}$	3136	1^{3136}	$\left(\phi_{56,30}\right)^2$
31	441	$(3A_1)''$	4	$\phi_{21,36}$	441	1^{441}	$\left(\phi_{21,36}\right)^2$
32	729	$2A_1$	3	$\phi_{27,37}$	721	1^{729}	$\left(\phi_{27,37}\right)^2$
33	49	A_1	2	$\phi_{7,46}$	49	1^{49}	$\left(\phi_{7,46}\right)^2$
34	1	0	1	$\phi_{1,63}$	1	1^1	$\left(\phi_{1,63}\right)^2$