Variations on a Formula of Selberg

OSU Representation Theory Seminar

September, 2005

1. Selberg's Formula

In 1944, in a short seven page paper in Norsk. Mat. Tidsakr, A. Selberg gives the following formula;

Theorem 1.1. *Let*

$$S_n\left(r,s,\kappa\right) \equiv \int_{[0,1]^n} \left(\prod_{i=1}^{n} \left(x_i\right)^{r-1} \left(1-x_i\right)^{s-1}\right) \left(\prod_{1\leq i<j\leq n} \left|x_i - x_j\right|^{2\kappa}\right) d\mathbf{x}$$

Then

$$S_n\left(\lambda_1,\lambda_2,\lambda\right) = \prod_{i=1}^{n} \frac{\Gamma\left(i\kappa+1\right)\Gamma\left(r+\left(i-1\right)\kappa\right)\Gamma\left(s+\left(i-1\right)\kappa\right)}{\Gamma\left(\kappa+1\right)\Gamma\left(r+s+2+\left(n-i-2\right)\kappa\right)}$$

Now the existence of some sort of closed formula for the Selberg integral is somewhat easy to understand, especially when λ is an even integer. Because in this case, by expanding the factor $\prod_{1\leq i<j\leq n} \left|x_i - x_j\right|^{\lambda}$ as a sum of monomials, one can rewrite the Selberg integral as a sum of product of beta integrals; i.e., integrals of the form

$$\beta\left(r,s\right) \equiv \int_0^1 x^{r-1}\left(1-x\right)^{s-1} dx = \frac{\Gamma\left(r\right)\Gamma\left(s\right)}{\Gamma\left(r+s\right)}$$

However, this simple algorithm for calculation does not explain the factorization of $S_n\left(\lambda_1,\lambda_2,\lambda\right)$, which is the truly remarkable part of Selberg's formula.

The Selberg formula lay dormant for about 40 years, until the 1980's when it began to crop up again in all sorts of places.

- In 1982, I.G. Macdonald uses the Selberg integral formula to prove a generalization of Dyson's conjecture: that the constant term in the Laurent polynomial

$$\prod_{\alpha\in R} \left(1-e^{\alpha}\right)^{k_\alpha}$$

  is

$$\prod_{\alpha\in R} \frac{\left(\left|\left\langle \rho_k, \alpha^\vee\right\rangle + k_\alpha + \frac{1}{2}k_{\alpha/2}\right|\right)!}{\left(\left|\left\langle \rho_k, \alpha^\vee\right\rangle + \frac{1}{2}k_{\alpha/2}\right|\right)!}$$

- Also in 1982, A. Koranyi uses the Selberg formula to compute the volumes of bounded symmetric domains.
- In 1987, K. Aomoto studies a slight generalization of the Selberg integral arising from work on Fock space representations of the Virasoro algebra. Here a connection with hypergeometric functions, specifically Jacobi polynomials is made.
- The Selberg integral formula also found its way (indirectly) into Bernstein degree calculations by H. Ochiai and his collaborators. (This is also the route by which I bumped into the Selberg integral.)

2. Digression: Symmetric Functions

At this point, I need to make a digression into the theory of symmetric functions, not only to highlight the special features of the Selberg integral that make it so prominent and point to natural generalizations, but, oddly enough, also to prepare to make a connection with generalized hypergeometric functions.

By a *symmetric polynomial* we mean a polynomial $p\left(\mathbf{x}_{(n)}\right) = p\left(x_1, \ldots, x_n\right)$ that is invariant under the natural action of the permutations group $\mathfrak{S}_n$; i.e.

$$p\left(x_1, \ldots, x_n\right) = p\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \qquad , \qquad \forall\, \sigma \in \mathfrak{S}_n$$

By a *symmetric function* we shall usually mean a function of, say n, variables that's similarly invariant under the natural action of $\mathfrak{S}_n$, although the same terminology is sometimes used to refer to a formal polynomial expression in an infinite number of variables, that's invariant under arbitary interchanges of variables.

There are several important bases for the symmetric polynomials in n variables, each parameterized by partitions of length n.

A partition λ of length n is simply a non-increasing list of non-negative integers; i.e., $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The weight $|\lambda|$ is the some of its parts; i.e., $|\lambda| = \sum_{i=1}^{n} \lambda_i$. There is a partial ordering of the set of partitions of weight w, called the *dominance partial ordering*, defined as follows

$$\mu \leq \lambda \quad \Longrightarrow \quad \sum_{i=1}^{m} \left(\lambda_i - \mu_i\right) \geq 0 \qquad , \qquad \text{for } m = 1, 2, \ldots, n$$

We start with the monomial symmetric functions. Let $\lambda = [\lambda_1, \ldots, \lambda_n]$ be a partition, and let $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ be the corresponding monomial. The *monomial symmetric function* m_λ is the sum of all distinct (monic) monomials that can be obtained from x^λ by permuting the x_i's. Every homogeneous symmetric polynomial of degree d can be uniquely expressed as a linear combination of the m_λ with $|\lambda| = d$.

The *power sum* symmetric polynomials are defined as follows. For each $r \geq 1$, let

$$p_r = m_{(r)} = \sum_{i=1}^{n} x_i^r$$

and then for any partition λ, set

$$p_\lambda\left(x\right) = p_{\lambda_1}\left(x\right) p_{\lambda_2}\left(x\right) \cdots p_{\lambda_n}\left(x\right)$$

The power sum symmetric functions $p_\lambda\left(x\right)$ with $|\lambda| = d$ provide another basis for the homogeneous symmetric polynomials of degree d. (In what follows, the power sum symmetric functions are only used to define a particular inner product for the symmetric polynomials; i.e., an inner product will be defined by specifying matrix entries with respect to the basis of power sum symmetric polynomials).

The *Schur polynomials* provide yet another basis. These can be defined as follows. For any partition λ of length n,

$$a_\lambda\left(x\right) = \det\left(x_i^{\lambda_j}\right)$$

The $a_\lambda\left(x\right)$ are obviously odd with respect to the action of the permutation group $\mathfrak{S}_n$; i.e. $a_\lambda\left(\sigma\left(x\right)\right) = sgn\left(\sigma\right) a_\lambda\left(x\right)$ for all $\sigma \in \mathfrak{S}_n$. However, it turns out that $a_\lambda = 0$ for and $\lambda < \delta \equiv [n-1, n-2, \ldots, 1, 0]$, and that, for every partition λ, $a_\delta\left(x\right)$ divides $a_{\lambda+\delta}\left(x\right)$. Indeed,

$$s_\lambda\left(x\right) \equiv \frac{a_{\lambda+\delta}\left(x\right)}{a_\delta\left(x\right)}$$

is a symmetric polynomial of degree $|\lambda|$. The polynomials $s_\lambda\left(x\right)$ are the Schur symmetric polynomials. The Schur polynomials $\{s_\lambda \mid |\lambda| = d\}$ provide another fundamental basis for the symmetric polynomials that are homogeneous of degree d. A special case that will be important to us latter on is

$$s_\delta\left(x\right) = \prod_{1 \leq i < j \leq n} \left(x_i + x_j\right)$$

where again $\delta = [n-1, n-2, \ldots, 1, 0]$. We note also the fact that $a_\delta(x)$ is just the Vandermonde determinant

$$a_\delta(x) = \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}^{n-1} & x_{n-1}^{n-2} & \cdots & x_{n-1} & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j) \quad .$$

The *zonal polynomials* Z_λ are defined as follows. Let $G = GL(n, \mathbb{R})$ and $K = O(n)$, the maximal compact subgroup of G. Let $P(G)$ be the space of polynomials on G, let $P(G/K)$ be the subpace of polynomials constant on the cosets G/K, and let $P(G,K)$ the subspace of polynomials constant on the double cosets $K\backslash G/K$. Let $P^m(G,K)$ be the subspace of homogeneous polynomials of degree m. It turns out that the polynomials $p \in P^{2m}(G,K)$ are determined by their values on the diagonal matrices, and that $p(g) = p^*\left(x_1^2, \ldots, x_n^2\right)$ where the x_i^2 are the eigenvalues of $g^T g$ and p^* is a symmetric polynomial in n-variables homogeneous of degree m. On the other hand, when one decomposes $P(G/K)$ into representations of G, the corresponding representations are parameterized by partitions λ, and within each irreducible summand the is a unique spherical vector $z_\lambda(g)$. The correspondence $z_\lambda(g) \to z_\lambda^*\left(x_1^2, \ldots, x_n^2\right)$ yields a symmetric polynomial, and in fact another partition-indexed basis for the symmetric polynomials.

Remark 2.1. A similar construction for the symmetric space $GL(n, \mathbb{C})/U(n)$ yields a basis for the symmetric polynomials that in fact coincides with the Schur basis.

Remark 2.2. In fact, the Schur polynomials can be uncovered in yet another way. It turns out that the Schur polynomial corresponding to a partition λ of length n, corresponds exactly to the Weyl character of the finite dimensional representation of SL_{n+1} corresponding to the partition λ; (i.e, the representation with highest weight

$$\lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n$$

as expressed in terms of the simple roots $\alpha_i = \varepsilon_i + \varepsilon_{i-1} \in \mathbb{R}^{n+1}$)[1]

Remark 2.3. And it turns out that are there are natural bases, indexed by partitions, associated with spherical polynomials on symmetric spaces for $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$ and $SL(n, \mathbb{H})$, and that the Jack symmetric polynomials $P_\lambda^{(\alpha)}$ for the cases $\alpha = 2,\ 1,$ and $\frac{1}{2}$ can also be as the spherical polynomials for, respectively, $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ and $GL(n, \mathbb{H})$.

Jack's symmetric functions $P_\lambda^{(\alpha)}$ are symmetric functions indexed by partitions and depending rationally on a parameter α, which interpolate between the Schur functions $s_\lambda(x)$ and the zonal polynomials $Z_\lambda(x)$. They are (uniquely) characterized by two properties

(i) $P_\lambda^{(\alpha)}(x) = m_\lambda(x) + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu(x)$; that is, the "leading term" of $P_\lambda^{(\alpha)}$ is the monomial symmetric functions m_λ and the remaining terms involve only monomial symmetric functions m_μ for which the partition index μ is less than λ with respect to the dominance ordering.

(ii) When one defines a scalar product on the vector space of homogeneous polynomials of degree d by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} z_\lambda$$
$$z_\lambda \equiv \prod_{r=1}^{n} \left(r^{m_\lambda(r)} \cdot m_\lambda(r)! \right)$$

[1] A little more precisely, the translation goes as follows

$$\chi_\lambda = \frac{\sum_{w \in W} sgn(w)\, e^{w(\lambda+\rho)}}{\sum_{w \in W} sgn(w)\, e^{w\rho}} \qquad \longleftrightarrow \qquad s_\lambda(x)$$

after changing variables to $x_i \equiv e^{\varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_{n+1})}$.

where $m_\lambda(r)$ is the number of times the integer r appears in λ, then

$$\left\langle P_\lambda^{(a)}, P_\mu^{(\alpha)} \right\rangle = 0 \qquad \text{if } \mu \neq \lambda \quad .$$

Remark 2.4. The question that should be forming in everybody's mind is: WTF! so many bases, can't this theory be formulated in a basis independent way. Well, of course, part of the reason for so many bases is that different natural bases for symmetric functions indexed by partitions crop up in many different contexts.

But with this complaint in mind, we point out that the initial interest in Jack symmetric functions lied precisely in the fact that by varying the parameter α one can interpolate between these various bases. (And latter developments have put the Jack symmetric polynomials inside other continuous families of symmetric polynomials).

2.1. **An alternative characterization of Jack symmetric polynomials.**

Theorem 2.5. *For each partition λ of length $\ell(n) \leq n$, there exists a polynomial $J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right) \equiv J_\lambda^{(\alpha)}(x_1,\ldots,x_n)$ with coefficients in $\mathbb{Q}(\alpha)$ satisfying the conditions*

- *(Triangularity) $J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right)$ is representable in the form*

(*) $$J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right) = \sum_{\mu\leq\lambda} v_{\lambda\mu} m_\lambda\left(\mathbf{x}_{(n)}\right) \qquad , \qquad v_{\lambda\mu} \in \mathbb{Q}(\alpha) \ ;$$

- *(Differential Equation).*

$$D(\alpha) J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right) = \beta_\lambda(\alpha) J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right)$$

where

$$D(\alpha) = \frac{\alpha}{2}\sum_{i=1}^n x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j} \frac{x_i^2}{x_i - x_j}\frac{\partial}{\partial x_i}$$
$$\beta_\lambda(\alpha) = \frac{\alpha}{2}\sum_i \lambda_i(\lambda_i - 1) - \sum_i (i-1)\lambda_i + (n-1)|\lambda|$$

- *(Normalization) If $|\lambda| = n$, then the coefficient of $J_{(1^n)}^{(\alpha)}\left(\mathbf{x}_{(n)}\right)$ in the expansion (*) is $n!$.*

Remark 2.6. This theorem is relatively easy to given the initial characterization of Jack symmetric functions (in terms of the inner product $\langle\cdot,\cdot\rangle_{(\alpha)}$; a straight-forward calculations reveal that, with respect to this inner product, $D(\alpha)$ is self-adjoint, and that $D(\alpha) J_\lambda^{(\alpha)}$ preserves the triangularity of the decomposition (*). Combined, these calculations infer that $D(\alpha)$ acts diagonally on the basis $\left\{J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right)\right\}$, and direct calculation of the action of $D(\alpha)$ on $n! m_{(1^n)}\left(\mathbf{x}_{(n)}\right)$, the term of (*) responsible for normalizing the $J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right)$, gives the eigenvalue $\beta_\lambda(\alpha)$.

Remark 2.7. The theorem also provides an easy way to establish the connection between zonal polynomials on symmetric spaces and the $J_\lambda^{(\alpha)}\left(\mathbf{x}_{(n)}\right)$; for the differential operator $D(\alpha)$ for $\alpha = 2, 1, \frac{1}{2}$ coincides with the Laplace-Beltrami operator for, respectively, $GL(n,\mathbb{R})/O(n)$, $GL(n,\mathbb{C})/U(n)$, and $GL(n,\mathbb{H})/U(n,\mathbb{H})$.

3. Back to the Selberg integral

Okay, now that we know a little about symmetric functions, let's re-examine the Selberg integral

$$S_n(r,s,\kappa) \equiv \int_{[0,1]^n} \left(\prod_{i=1}^n (x_i)^{r-1}(1-x_i)^{s-1}\right)\left(\prod_{1\leq i<j\leq n} |x_i - x_j|^{2\kappa}\right) d\mathbf{x}$$

We now note the following:

- The integrand is a symmetric function:
  In fact, the factor $\prod_{1\leq i<j\leq n}|x_i - x_j|$ is just the Vandermonde determinant
$$a_\delta(x) = \det\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}^{n-1} & x_{n-1}^{n-2} & \cdots & x_{n-1} & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix}$$
  and the factor $\prod_{i=1}^n x_i$ is just $m_{(1^n)}(\mathbf{x})$. The factor $\prod_{i=1}^n(1-x_i)$ is also symmetric.
- The region of integration is also $\mathfrak{S}_n$ invariant and so the crux of the problem is to compute
$$S_n(r,s,\kappa) = n!\int_{\Omega_n}\left(\prod_{i=1}^n (x_i)^{r-1}(1-x_i)^{s-1}\right)\left(\prod_{1\leq i<j\leq n}|x_i - x_j|^{2\kappa}\right)$$
  where
$$\Omega_n = \{\mathbf{x}\in\mathbb{R}^n \mid 1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$
  invariant under the change of variables $x_i \to 1 - x_i$. In fact, it easy to see that
$$S_n(r,s,\kappa) = (-1)^n S_n(s,r,\kappa)$$

Immediately, several two natural generalizations/modifications spring to mind.

- Generalize the integrand to include more general symmetric functions; say
$$I_{n,\lambda,r,s,k} \equiv \int_{[0,1]^n} P_\lambda^{(\alpha)}(\mathbf{x})\left(\prod_{i=1}^n (x_i)^{r-1}(1-x_i)^{s-1}\right)\left(\prod_{1\leq i<j\leq n}|x_i - x_j|^{2\kappa}\right)d\mathbf{x}$$
  or
- Change the domain of integration to another fundamental region for $\mathfrak{S}_n$.

Now in the application we originally had in mind, the computation of the Bernstein degree for certain unipotent representations, both sorts of modifications had to be dealt with. However, there was nothing in the literature to deal with our particular case, which was
$$\int_{\mathcal{C}_n}\left(\prod_{i=1}^n x_i\right)^p\left(\prod_{1\leq i<j\leq n}\left|x_i^2 - x_j^2\right|^d\right)d\mathbf{x} = \int_{\mathcal{C}_n}\left(\prod_{i=1}^n x_i\right)^p (s_\delta(\mathbf{x}))^d\, a_\delta(\mathbf{x})^d\, d\mathbf{x}$$
where
$$\mathcal{C}_n = \left\{\mathbf{x}\in\mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\ ,\ \sum_{i=1}^n x_i \leq 1\right\}$$
The case $d = 1$ is particularly gruesome since the integrand is actually only invariant under the alternating group (the subgroup of even elements of $\mathfrak{S}_n$).

Okay, I must admit that, so far, this looks like a lot a bru-ha-ha over a Calculus II problem. So let me introduce another element at play here.

4. Generalized Hypergeometric Functions

4.1. **Classical Hypergeometric Functions.** Okay, first of all what are hypergeometric functions (of one variable). Well, the most expedient way to introduce hypergeometric equations is by their power series expansions. Let
$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

and then hypergeometric function ${}_qF_p\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right)$ is

$$ {}_qF_p\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right)\equiv\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\cdots\left(a_p\right)_n}{\left(b_1\right)_n\cdots\left(b_q\right)_n n!}x^n $$

The same series can also be derived as the formal power series solution to

$$ \left[x\frac{d}{dx}\left(x\frac{d}{dx}+b_1-1\right)\cdots\left(x\frac{d}{dx}+b_q-1\right)-x\left(x\frac{d}{dx}+a_1\right)\cdots\left(x\frac{d}{dx}+a_p\right)\right]y=0 $$

But to understand their importance, it's probable best to cite some of their various incarnations:

- Legendre Functions; $P_\nu\left(x\right)={}_2F_1\left(-\nu,\nu+1;1;\frac{1-x}{2}\right)$
- Gegenbauer polynomials; $C_n^\lambda\left(x\right)=\frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(2\lambda)}\ {}_2F_1\left(2\lambda+n,-n;\lambda+\frac{1}{2},\frac{1-x}{2}\right)$
- Bessel functions: $J_\alpha\left(x\right)=\frac{(x/2)^\alpha}{\Gamma(\alpha+1)}e^{-ix}\ {}_1F_1\left(\alpha+\frac{1}{2};2\alpha+1;2ix\right)$
- Whittaker functions; (a linear combination of) $W_{\lambda,\pm\mu}\left(x\right)=x^{\pm\mu+\frac{1}{2}}e^{-x/2}\ {}_1F_1\left(\pm\mu-\lambda,2\left|\mu\right|+1;x\right)$
- Hermite polynomials: $H_{2n}\left(x\right)=\left(-1\right)^n\frac{(2n)!}{n!}\ {}_1F_1\left(-n;\frac{1}{2};x^2\right)\quad,\quad H_{2n+1}=\left(-1\right)^n\frac{2(2n+1)!}{n!}x\ {}_1F_1\left(-n;\frac{3}{2};x^2\right)$
- Laguerre polynomials: $L_n^\alpha\left(s\right)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\ {}_1F_1\left(-n;\alpha+1;x\right)$

Even relatively elementary functions have simple expressions in terms of hypergeometric functions; for example,

$$ \begin{aligned} \log\left(1+x\right)&=x\ {}_2F_1\left(1,1;2;-x\right)\\ \tan^{-1}\left(x\right)&=x\ {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)\\ e^x&={}_0F_0\left(-;-;x\right) \end{aligned} $$

etc..

In summary and understatement, classical hypergeometric functions provide a nice bit of mathematical infrastructure.

4.2. **Generalized Hypergeometric Functions.** Generalized hypergeometric functions were first introduced by Herz in the 50's by means of a multivariant Laplace transform. In the one variable case, one can use the Laplace transform to build up the entire array of hypergeometric functions inductively, by starting with ${}_0F_0\left(x\right)\equiv e^x$.

Hypergeometric functions of matrix argument have been studied since the 60's by statisticians (most notably, James) where they were introduced as power series of the form

$$ {}_pF_q\left(a_1;\ldots,a_p;b_1,\ldots,b_q;\mathbf{t}_{(n)}\right)=\sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{\left[a_1\right]_\lambda\cdots\left[a_p\right]_\lambda}{\left[b_1\right]_\lambda\cdots\left[b_q\right]_\lambda d!}Z_\lambda\left(\mathbf{t}_{(n)}\right) $$

Here the $Z_\lambda\left(\mathbf{t}_{(n)}\right)$ is (up to normalization) the zonal polynmomial in n variables $t_1,\ldots,t_n$ corresponding to a partition $\lambda=\left(\lambda_1,\ldots,\lambda_n\right)$, and $\left[a\right]_\lambda$ is the generalized Pochhammer symbol:

$$ \left[a\right]_\lambda=\prod_{i=1}^{n}\left(a-\frac{1}{2}\left(i-1\right)\right)_{\lambda_i}\equiv\prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}\left(a-\frac{1}{2}\left(i-1\right)+j-1\right) $$

Of course, the zonal polynomials also have a group theoretical interpretation, they are the spherical polynomials on $GL\left(n,\mathbb{R}\right)/0\left(n\right)$ and generalize to spherical polynomials on other bounded symmetric domains. And as such, that's how notions of generalized hypergeometric have gained some some prominence in analytic representation theory.

However, a more encompassing generalization is to replace the expansion in terms of zonal polynomials by an expansion in terms of Jack symmetric functions $J_{\lambda}^{(\alpha)}$. One can then obtain hypergeometric functions on $GL(n,\mathbb{R})/O((n))$, $GL(n,\mathbb{C})/U(n)$, and $GL(n,\mathbb{H})/U(n,\mathbb{H})$, by simply setting $\alpha = 2, 1$ or $\frac{1}{2}$.

Thus, Kaneko defines

$$
{}_pF_{qp}^{(\alpha)}\left(a_1;\dots,a_p;b_1,\dots,b_q;\mathbf{t}_{(n)}\right) = \sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{[a_1]_{\lambda}^{(\alpha)}\cdots[a_p]_{\lambda}^{(\alpha)}}{[b_1]_{\lambda}^{(\alpha)}\cdots[b_q]_{\lambda}^{(\alpha)}d!}J_{\lambda}^{(\alpha)}\left(\mathbf{t}_{(n)}\right)
$$

where, now the generalized Pochhammer symbol $[a]_{\lambda}^{(\alpha)}$ corresponding to a partition λ and parameter α is

$$
[a]_{\lambda}^{(\alpha)} = \prod_{i=1}^{\ell(\lambda)}\left(a - \frac{1}{\alpha}(i-1)\right)_{\lambda_i}
$$

What's particular nice about Kaneko's formulation is that he shows that the hypergeometric function ${}_2F_1^{(a)}(a,b;c,\mathbf{t})$ is the unique symmetric formal power series satisfying

$$
\begin{aligned}
0 &= t_i(1-t_i)\frac{\partial^2 F}{\partial t^2} + \left(c - \frac{1}{\alpha}(n-1) - \left(a+b+1-\frac{1}{\alpha}(m-1)\right)t_i\right)\frac{\partial F}{\partial t_i} \\
&+ \frac{1}{\alpha}\left(\sum_{\substack{i,j\\ i\neq j}}\frac{t_i(1-t_i)}{t_i-t_j}\frac{\partial F}{\partial t_i} - \sum_{\substack{i,j\\ i\neq j}}\frac{t_j(1-t_j)}{t_i-t_j}\frac{\partial F}{\partial t_j}\right) - abF
\end{aligned}
$$

and

$$
F(0) = 1
$$

5. Kaneko's result

Next time I intend to sketch how Kaneko uses this apparatus to obtain the following result

Theorem 5.1. *Let*

$$
S_{n,m}\left(\lambda_1,\lambda_2,\lambda;\mathbf{t}_{(m)}\right) \equiv \int_{[0,1]^n}\left(\prod_{1\le i\le n}\prod_{1\le k\le m}(x_i-t_k)(x_i)^{\lambda_1}(1-x_i)^{\lambda_2}\right)\left(\prod_{1\le i<j\le n}|x_i-x_j|\right)^{\lambda}d\mathbf{x}_{(n)}
$$

Then

$$
S_{n,m}\left(\lambda_1,\lambda_2,\lambda;\mathbf{t}_{(m)}\right) = C_1\ {}_2F_1^{\lambda/2}\left(-n,\frac{2}{\lambda}(\lambda_1+\lambda_2+m+1)+n-1;\frac{2}{\lambda}(\lambda_1+m);\mathbf{t}_{(n)}\right)
$$

where

$$
C_1 = S_{n,0}(\lambda_1+m,\lambda_2,\lambda)
$$

6. (Intermission)