Variations on a Formula of Selberg

Part II: Selberg Integrals and Hypergeometric Functions à la Kaneko

OSU Representation Theory Seminar

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1. The plot so far

Last time I introduced

- **generalized Selberg integrals** as integrals of the form

$$I_{n,\lambda,r,s,\kappa} = \int_{\Omega_n} \Phi(\mathbf{x}) \left(\prod_{i=1}^{n} x_i^{r-1} (1-x_i)^{s-1} \right) \left(\prod_{1\le i<j\le n} |x_i - x_j|^{2\kappa} \right) dx_1 \cdots dx_n \tag{1}$$

  Φ being some symmetric polynomials on $\mathbb{R}^n$ and Ω_n some fundamental domain for the action the symmetric group $\mathfrak{S}_n$.
- **Jack symmetric functions** $J_\lambda^{(\alpha)}$ as a particular basis of symmetric polynomials uniquely defined by the requirements
  - $\left\langle J_\mu^{(\alpha)}, J_\lambda^{(\alpha)} \right\rangle_\alpha = 0$ if $\lambda \neq \mu$ (orthogonality) where the inner product $\langle \cdot,\cdot \rangle_\alpha$ is defined by

$$\begin{aligned} \langle p_\lambda, p_\mu \rangle &= \delta_{\lambda,\mu} \alpha^{|\lambda|} z_\lambda \\ z_{(1^{m_1} 2^{m_2} \cdots)} &\equiv \prod_{i=1}^{n} i^{m_i} (m_i)! \\ &= \text{order of the centralizer in } \mathfrak{S}_n \text{ of a cycle of type } \lambda \end{aligned}$$

  - $J_\lambda^{(\alpha)} = \sum_{\mu\le\lambda} v_{\lambda\mu} m_\mu$ (triagularity)
  - If $|\lambda| = d$, then $v_{\lambda,(1^d)} = d!$

  The Jack symmetric functions are eigenfunctions of the following differential operator

$$D(\alpha) = \frac{\alpha}{2} \sum_{i=1}^{n} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i} \tag{2}$$

  with eigenvalue

$$e_\lambda(\alpha) = \frac{\alpha}{2} \sum_{i=1}^{m} \lambda_i (\lambda_i - 1) - \sum_{i=1}^{m} (i-1)\lambda_i + (n-1)|\lambda| \tag{3}$$

- **Generalized hypergeometric functions**

$${}_pF_q^{(a)}(a_1,\ldots,a_p; b_1,\ldots,b_q; \mathbf{t}) = \sum_{d=0}^{\infty} \sum_{|\lambda|=d} \frac{[a_1]_\lambda^{(\alpha)} \cdots [a_p]_\lambda^{(\alpha)}}{[b_1]_\lambda^{(\alpha)} \cdots [b_q]_\lambda^{(\alpha)}\, d!} C_\lambda^{(\alpha)}(\mathbf{t}) \tag{4}$$

  where

$$\begin{aligned} C_\lambda^{(\alpha)}(\mathbf{t}) &= \alpha^{|\lambda|} |\lambda|! J_\lambda^{(\alpha)}(\mathbf{t}) \\ [a]_\lambda^{(\alpha)} &= \prod_{i=1}^{\ell(\lambda)} \left(a - \frac{1}{\alpha}(i-1) \right)_{\lambda_i} \end{aligned}$$

2. Kaneko's Generalized Selberg Integral

In this talk we shall be considering the generalized Selberg integral studied by Kaneko

(6) $$\begin{aligned} & S_{n,m}\left(\lambda_1,\lambda_2,\lambda,\mu;t_1,\ldots,t_m\right) \\ & = \int_{[0,1]^n}\left(\prod_{\substack{l\leq i\leq n\\ i\leq k\leq m}}\left(x_i-t_k\right)^{\mu}\right)\left(\prod_{1\leq i\leq n}x_i^{\lambda_1}\left(1-x_i\right)^{\lambda_2}\right)\left(\prod_{1\leq i<j\leq n}\left|x_i-x_j\right|^{\lambda}\right)dx_1\cdots dx_n \end{aligned}$$

for which the original Selberg integral corresponds to the special case of $S_{n,0}\left(\lambda_1,\lambda_2,\lambda,0;\mathbf{0}\right)$.

3. Holonomic System for $S_{n,m}$

Let us denote by Φ the integrand of (6):

$$\Phi=\left(\prod_{\substack{l\leq i\leq n\\ i\leq k\leq m}}\left(x_i-t_k\right)^{\mu}\right)\left(\prod_{1\leq i\leq n}x_i^{\lambda_1}\left(1-x_i\right)^{\lambda_2}\right)\left(\prod_{1\leq i<j\leq n}\left|x_i-x_j\right|^{\lambda}\right)$$

let ω be the logarithmic 1-form[1]

$$\omega=d\log\Phi$$

and let ∇_{ω} be the covariant differentiation defined by

$$\nabla_{\alpha}\varphi=d\varphi+\omega\wedge\varphi$$

for any smooth $(n-1)$-form φ. One has

$$\begin{aligned} d\left(\Phi\varphi\right) &= \left(d\Phi\right)\wedge\varphi+\Phi\left(d\varphi\right) \\ &= \Phi\left(d\varphi+\frac{1}{\Phi}\left(d\Phi\right)\wedge\varphi\right) \\ &= \Phi\left(\nabla_{\omega}\varphi\right) \end{aligned}$$

and so by Stokes theorem, and the fact that Φ vanishes on each face of the cube $[0,1]^n$,

(*) $$\int_{[0,1]^n}\Phi\nabla_{\omega}\varphi=\int_{[0,1]^n}d\left(\Phi\varphi\right)=\int_{\partial([01,]^n)}\Phi\varphi=0$$

as long as the left hand side exists.

Kaneko utilizes the identity (*) for three easy choices of $(n-1)$-forms φ and to provide identities certain derivatives of $I_{n,m}$. Let us denote by ${}^*dx_i$ the $(n-1)$-form

$${}^*dx_i=\left(-1\right)^{i-1}dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_n$$

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[1] Explicitly,

$$\begin{aligned} \omega &= d\log\left(\left(\prod_{i=1}^{n}x_i^{\lambda_1}\left(1-x_i\right)^{\lambda_2}\right)\left(\prod_{1\leq i<j\leq n}\left|x_i-x_j\right|^{\lambda}\right)\left(\prod_{\substack{1\leq i\leq n\\ 1\leq k\leq m}}\left(x_i-t_k\right)^{\mu}\right)\right) \\ &= \frac{1}{\Phi}\sum_{i=1}^{n}\left(\frac{\lambda_1}{x_i}\Phi+\frac{\lambda_2}{1-x_i}\Phi+\sum_{j=1}^{i-1}\frac{-\lambda}{x_j-x_i}\Phi+\sum_{j=i+1}^{n}\frac{\lambda}{x_i-x_j}\Phi+\sum_{1\leq k\leq m}\frac{\mu}{x_i-t_k}\Phi\right)dx_i \end{aligned}$$

and put

$$\varphi_0 = \sum_{i=1}^{n} {}^*dx_i$$

$$\varphi_1 = \sum_{i=1}^{n} x_i \ {}^*dx_i$$

$$\psi_k = \sum_{i=1}^{n} (x_i - t_k)^{-1} \ {}^*dx_i \qquad , \qquad 1 \le k \le m$$

The covariant differentiations of these forms are

$$\nabla_\omega \varphi_0 = \left[\lambda_1 \sum_{i=1}^{n} x_i^{-1} - \lambda_2 \sum_{i=1}^{n} (1 - x_i)^{-1} + \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} (x_i - t_k)^{-1} \right] \theta \tag{3}$$

$$\nabla_\omega \varphi_1 = \left[n \left(1 + \lambda_1 + \lambda_2 + m\mu + \frac{n-1}{2} \lambda \right) - \lambda_2 \sum_{i=1}^{n} (1 - x_i)^{-1} + \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} \frac{t_k}{x_i - t_k} \right] \theta \tag{4}$$

$$\begin{aligned} \nabla_\omega \psi_k = \Bigg[& (\mu - 1) \sum_{i=1}^{n} (x_i - t_k)^{-2} - \lambda \sum_{1 \le i < j \le n} ((x_i - t_k)(x_j - t_k)) + \lambda_1 t_k^{-1} \\ & \cdot \left(\sum_{i=1}^{n} (x_i - t_k)^{-1} - \sum_{i=1}^{n} x_i^{-1} \right) - \lambda_2 (1 - t_k)^{-1} \left(\sum_{i=1}^{n} (1 - x_i)^{-1} + \sum_{i=1}^{n} (x_i - t_k)^{-1} \right) \\ & + \mu \sum_{\substack{l=1 \\ l \ne m}}^{m} (t_k - t_l)^{-1} \left(\sum_{i=1}^{n} (x_i - t_k)^{-1} - \sum_{i=1}^{n} (x_i - t_l)^{-1} \right) \Bigg] \theta \end{aligned} \tag{5}$$

where θ denotes the volume n-form: $\theta = dx_1 \wedge \cdots \wedge dx_n$. For n-forms ξ, η, we write $\xi \sim \eta$ if $\xi - \eta = \nabla_\omega \varphi$ for some $(n-1)$-form φ. It follows from (3) and (4) that

$$\nabla_\omega (\varphi_0 - \varphi_1) = \left[\lambda_1 \sum_{i=1}^{n} x_i^{-1} - n \left(1 + \lambda_1 + \lambda_2 + m\mu + \frac{n-1}{2} \lambda \right) + \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} \frac{1 - t_k}{x_i - t_k} \right] \theta$$

or

$$\left[\lambda_1 \sum_{i=1}^{n} x_i^{-1} \right] \theta = \left[n \left(1 + \lambda_1 + \lambda_2 + m\mu + \frac{n-1}{2} \lambda \right) - \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} \frac{1 - t_k}{x_i - t_k} \right] \theta + \nabla_\omega (\varphi_0 - \varphi_1)$$

$$\left[\lambda_2 \sum_{i=1}^{n} (1 - x_i)^{-1} \right] \theta \sim \left[n \left(1 + \lambda_1 + \lambda_2 + m\mu + \frac{n-1}{2} \lambda \right) + \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} \frac{t_k}{x_i - t_k} \right] \theta$$

And (4) leads directly to

$$\lambda_2 \sum_{i=1}^{n} (1 - x_i)^{-1} = \left[n \left(1 + \lambda_1 + \lambda_2 + m\mu + \frac{n-1}{2} \lambda \right) + \mu \sum_{\substack{1 \le i \le n \\ 1 \le k \le m}} \frac{t_k}{x_i - t_k} \right] \theta - \nabla_\omega \varphi_1$$

Substituting these into (5), we obtain

$$\begin{aligned}\nabla_\omega \psi_k \sim \Bigg[& (\mu-1)\sum_{i=1}^{n}(x_i-t_k)^{-2} - \lambda \sum_{1\le i<j\le n}((x_i-t_k)(x_j-t_k))^{-1} \\ & + \left(\lambda_1 t_k^{-1} - \lambda_2 (1-t_k)^{-1}\right)\left(\sum_{i=1}^{n}(x_i-t_k)^{-1}\right) \\ & - t_k^{-1}\left(n\left(1+\lambda_1+\lambda_2+m\mu+\frac{n-1}{2}\lambda\right) - \mu \sum_{\substack{1\le i\le n\\ 1\le l\le m}} \frac{1-t_l}{x_i-t_l}\right) \\ & - (1-t_k)^{-1}\left(n\left(1+\lambda_1+\lambda_2+m\mu+\frac{n-1}{2}\lambda\right) + \mu \sum_{\substack{1\le i\le n\\ 1\le l\le m}} \frac{t_l}{x_i-t_l}\right) \\ & + \mu \sum_{\substack{l=1\\ l\neq k}}^{n}(t_k-t_l)^{-1}\left(\sum_{i=1}^{n}(x_i-t_k)^{-1} - \sum_{i=1}^{n}(x_i-t_l)^{-1}\right)\Bigg]\theta \end{aligned} \tag{(**)}$$

On the other hand, once can easily show that

$$\frac{\partial S_{n,m}(t)}{\partial t_k} = -\mu \int_{[0,1]^n} \Phi \left[\sum_{i=1}^{n}(x_i-t_k)^{-1}\right]\theta \tag{7}$$

$$\frac{\partial^2 S_{n,m}(t)}{\partial t_k^2} = \int_{[0,1]^n} \Phi \left[\left(\mu^2-\mu\right)\sum_{i=1}^{n}(x_i-t_k)^{-2} + 2\mu^2 \sum_{1\le i<j\le n}((x_i-t_k)(x_j-t_k))^{-1}\right]\theta \tag{8}$$

Suppose now that the ratio $\left(\mu^2-\mu\right)/2\mu^2$ equals $(\mu-1)/(-\lambda)$; i.e., $\mu=1$ or $\mu=-\lambda/2$. From (*) we have

$$0 = \int_{[0,1]^n} \Phi \nabla_\omega \psi_k$$

by (8), if we use (**) to expand the right hand side, the first two sums add up to a constant multiple of $\partial^2 S_{n,m}(t)/\partial t_k^2$. Hence, by virtue of (7) and (8), taking ψ_k for φ of (2) yields a partial differential equation of $S_{n,m}(t)$ for each k. Moreover, its principle part contains only $\partial^2 S_{n,m}(t)/\partial t_k^2$. We thus have.

Theorem 3.1. *Assume $\mu=1$ or $\mu=-\lambda/2$. Then $S_{n,m}(\lambda_1,\lambda_2,\lambda,\mu;t)$ satisfies the following holonomic system*

$$\begin{aligned} 0 = {} & t_i(1-t_i)\frac{\partial F}{\partial t_i^2} + \left\{c - \frac{1}{\alpha}(m-1) - \left(a+b+1-\frac{1}{\alpha}(m-1)\right)t_i\right\}\frac{\partial F}{\partial t_i} - abF \\ & + \frac{1}{\alpha}\left\{\sum_{\substack{j=1\\ j\neq i}} \frac{t_i(1-t_i)}{t_i-t_j}\frac{\partial F}{\partial t_j} - \sum_{\substack{j=1\\ j\neq i}} \frac{t_j(1-t_j)}{t_i-t_j}\frac{\partial F}{\partial t_j}\right\}, \qquad i=1,\ldots,m \end{aligned} \tag{9}$$

where, if $\mu=1$,

$$\begin{aligned} \alpha &= \lambda/2 \\ a &= -n \\ b &= (2/\lambda)(\lambda_1+\lambda_2+m+1) \\ c &= (2/\lambda)(\lambda_1+m) \end{aligned}$$

and if $\mu = -\lambda/2$

$$\begin{aligned} \alpha &= \lambda/2 \\ a &= (\lambda/2)\, n \\ b &= -\left(\lambda_1 + \lambda_2 + 1\right) + (\lambda/2)\,(m - n + 1) \\ c &= -\lambda_1 + (\lambda/2)\, m \end{aligned}$$

4. Hypergeometric Solution of the Holonomic System

Theorem 4.1. ${}_2F_1^{(\alpha)}\left(a, b; c; \mathbf{t}\right)$ *is the unique solution to each of the* m *differereential equations in the system (9) subject to the following conditions:*

- $F(\mathbf{t})$ *is a symmetric function of* $t_1, \ldots, t_m$
- $F(\mathbf{t})$ *is analytic at the origin with* $F(\mathbf{0}) = 1$

Sketch of Proof.

Uniqueness:

Noting that a symmetric analytic solution of (9) must be expressible as a power series $\mathbb{C}\left[\left[r_1, \ldots, r_n\right]\right]$ where the r_i are some rational basis for the symmetric polynomials, Kaneko changes variables $t_i \to r_i(\mathbf{t})$ where $r_i(\mathbf{t})$ is the i^{th} elementary symmetric polynomial in $\mathbf{t}$ and makes an ansatz

$$F(\mathbf{t}) = \sum_{\lambda \in \mathcal{P}(m)} a_\mu r_\mu(\mathbf{t}) \qquad , \qquad r_\mu(\mathbf{t}) = r_{\mu_1}(\mathbf{t}) \cdots r_{\mu_m}(\mathbf{t}) \qquad , \qquad a_\mu \in \mathbb{C}$$

and then shows that their is a total ordering $\overset{R}{<}$ of the partitions λ for which recursion relations for the coefficients a_μ take the form

$$a_\lambda = \text{sum of } a_\mu \text{ with } \mu \overset{R}{<} \lambda$$

Solution in terms of hypergeometric functions:

Summing the equations in (9) one sees that a symmetric solution of (9) must satisfy

$$(9) \qquad 0 = \sum_{i=1}^{m} t_i\left(1 - t_i\right) \frac{\partial F}{\partial t_i^2} + \sum_{i=1}^{m} \left\{ c - \frac{1}{\alpha}(m-1) - \left(a + b + 1 - \frac{1}{\alpha}(m-1) \right) t_i \right\} \frac{\partial F}{\partial t_i} - abF$$
$$+ \frac{1}{\alpha} \sum_{i=1}^{m} \left\{ \sum_{\substack{j=1 \\ j \neq i}} \frac{t_i\left(1 - t_i\right)}{t_i - t_j} \frac{\partial F}{\partial t_j} - \sum_{\substack{j=1 \\ j \neq i}} \frac{t_j\left(1 - t_j\right)}{t_i - t_j} \frac{\partial F}{\partial t_j} \right\}$$

Kaneko then establishes certain derivative identities for Jack symmetric functions[2] which allow him to conclude that if one sets

$$F(\mathbf{x}) = \sum_{d=0}^{\infty} \sum_{|\lambda|=d} c_\lambda C_\lambda^{(\alpha)}(\mathbf{x}) \qquad , \qquad C_\lambda^{(\alpha)}(\mathbf{x}) \equiv \frac{\alpha^{|\lambda|}\,|\lambda|!}{\langle J_\lambda, J_\lambda\rangle_\alpha} J_\lambda^{(\alpha)}(\mathbf{x})$$

and chooses the coefficients c_λ as

$$c_\lambda = \frac{[a]_\lambda^{(\alpha)}\,[b]_\lambda^{(\alpha)}}{[c]_\lambda^{(\alpha)}\,|\lambda|!}$$

then $F(\mathbf{x})$ satisfies (9) identically.

5. The main result

Well, the main result is now sorta obvious as the Kaneko's generalized Selberg integrals satisfies a certain holonomic system of PDEs for which the generalized hypergeometric function is the unique symmetric analytic solution satisfying $F(0) = 1$. The only thing left is to verify that the generalized Selberg integral is analytic at the origin and to determine appropriate multiplicative constant. However, the case when $t = 0$ corresponds to the original Selberg integral, which is known.

In addition, Kaneko gives a sort of Kummer formula allowing a slight extension of the obvious result.

Proposition 5.1. *If $F(t_1, \ldots, t_m)$ is a solution of the system (9), then $(t_1 \cdots t_m)^{-a} F\left(t_1^{-1}, \ldots, t_m^{-1}\right)$ is also a solution of the system obtained from (9) by replacing b by $a-c+1+(m-1)/\alpha$ and c by $a-b+1+(m-1)/\alpha$.*

Thus, Kaneko obtains

Theorem 5.2. *Let*

$$S_{n,m}\left(\lambda_1, \lambda_2, \lambda, \mu; \mathbf{t}_{(m)}\right) = \int_{[0,1]^n} \left(\prod_{\substack{1\le i\le n \\ 1\le k\le m}} (x_i - t_k)\right)^{\mu} \left(\prod_{i=1}^{n} x_i^{\lambda_1}(1-x_i)^{\lambda_2}\right) \left(\prod_{1\le i<j\le n} |x_i - x_j|^{\lambda}\right) d\mathbf{x}_{(n)}$$

Then

$$S_{n,m}\left(\lambda_1, \lambda_2, \lambda, 1; \mathbf{t}_{(m)}\right) = C_1\ {}_2F_1^{\lambda/2}\left(-n, \frac{2}{\lambda}(\lambda_1 + \lambda_2 + m + 1) = n - 1; \frac{2}{\lambda}(\lambda_1 + m); \mathbf{t}_{(m)}\right)$$

where

$$C_1 = S_{n,0}(\lambda_1 + m, \lambda_2, \lambda)$$

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[2] These identities are relatively straight-forward, the first is just the fact that the Jack symmetric functions are eigenfunctions of

$$D(\alpha) = \frac{\alpha}{2}\sum_{i=1}^{n} x_i^2 \frac{\partial}{\partial x_i^2} + \sum_{i\ne j} \frac{x_i}{x_i - x_j}\frac{\partial}{\partial x_i}$$

The second that they are eigenfunctions of the Euler operator

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

And the third gives an expression for

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} J_\lambda(\mathbf{x})$$

in terms of generalized binomial coefficients.

Moreover,

$$\int_{[0,1]^n} \left(\prod_{\substack{1\le i\le n \\ 1\le k\le m}} (1 - x_i t_k) \right)^{-\lambda/2} \left(\prod_{i=1}^{n} x_i^{\lambda_1} (1-x_i)^{\lambda_2} \right) \left(\prod_{1\le i<j\le n} |x_i - x_j|^{\lambda} \right) d\mathbf{x}_{(n)}$$
$$= C_2 \; {}_2F_1^{2/\lambda} \left(\frac{\lambda}{2} n, \frac{\lambda}{2} (n-1) + \lambda_1; \lambda (n-1) + \lambda_1 + \lambda_2 + 2; \mathbf{t}_{(m)} \right)$$

where

$$C_2 = S_{n,0} (\lambda_1, \lambda_2, \lambda)$$

6. Dessert

Corollary 6.1. *Let μ be a partition and set*

$$I_\mu \equiv \int_{[0,1]^n} J_\mu^{2/\lambda} \left(\mathbf{x}_{(n)} \right) \left(\prod_{i=1}^{n} x_i^{\lambda_1} (1-x_i)^{\lambda_2} \right) \left(\prod_{1\le i<j\le n} |x_i - x_j|^{\lambda} \right) d\mathbf{x}_{(n)}$$

Then

$$I_\mu = J_\mu^{(2/\lambda)} \left(\mathbf{1}_{(n)} \right) \prod_{i=1}^{n} \frac{\Gamma (i\lambda/2 + 1) \Gamma (\mu_i + \lambda_1 + (n-i) \lambda/2 + 1) \Gamma (\lambda_2 + (n-i) \lambda./2 + 1)}{\Gamma (\lambda/2 + 1) \Gamma (\mu_i + \lambda_1 + \lambda_2 + (2n - i = 1) \lambda/2 + 2)}$$

This is proved by simply plugging the generalized Cauchy identity

$$\prod_{\substack{i\le i\le n \\ 1\le k\le n}} (1 - x_i t_k)^{-1/\alpha} = \sum_{\nu} J_\nu^{(\alpha)} \left(\mathbf{x}_{(n)} \right) J_\nu^{(\alpha)} \left(\mathbf{t}_{(n)} \right) \frac{1}{\langle J_\lambda, J_\lambda \rangle_\alpha}$$

into the integrand in () and then equating the coefficients of $J_\mu^{(2/\lambda)} \left(\mathbf{t}_{(n)} \right)$ that occur on both sides (recall that ${}_2F_1^{2/\lambda} \left(\mathbf{t}_{(n)} \right)$ is defined as an expansion in the $J_\mu^{(2/\lambda)} \left(\mathbf{t}_{(n)} \right)$.

Thus, in just a couple lines one proves a famous conjecture of Macdonald, latter proved by Kadell. (At the time of the conjecture it was known that there existed a family of symmetric functions with such a closed integral formula, Macdonald conjectured that this family would be the Jack symmetric functions, and Kadell proved it. This development took place from around 1986- 1996).