Bernstein degree computations and Selberg type integrals OSU Representation Theory Seminar March 1, 2006

1. Bernstein degrees of multiplicity free $(\mathfrak{g}, K)$-modules

In previous seminars I've talked about the Selberg type integral that arises in the computation of the Bernstein degrees of certain small unitary representations associated with simple non-Euclidean Jordan algebras.

In this seminar I'll instead introduce the same family of integrals as an extension of the results of Kato and Ochiai to the case where the restricted root systems are of type B_n or C_n. (Kato and Ochiai provide an explicit evaluation of the Bernstein degree integrals only for the hermitian symmetric case (type A_n case)).

Let G_0 be a real reductive Lie group, let K_0 a maximal compact subgroup of G_0 with associated Cartan involution θ, set $\mathfrak{g} = Lie\,(G_0)_{\mathbb{C}}$, $\mathfrak{k} = Lie\,(K_0)_{\mathbb{C}}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Let K let be the connected simply connected reductive complex algebraic group with Lie algebra $\mathfrak{k}$ and let V be a $(\mathfrak{g}, K)$-module with the following properties:

- V is multiplicity free; i.e., each K-type in V is of multiplicity one.
- There exists linearly independent weights $\varphi_1, \ldots, \varphi_n \in \mathfrak{t}^*$ such that
$$V = \bigoplus_{\mu \in \Lambda(V)} V_{\lambda+\mu}$$
where V_λ is the irreducible finite-dimensional K-module with highest weight λ with respect to some Cartan algebra $\mathfrak{t}$ of $\mathfrak{k}$ and
$$\Lambda\,(V) = \left\{\mu = m_1\varphi_1 + \cdots m_q\varphi_q \mid m_1 \geq m_2 \geq \cdots \geq m_n \quad , \quad m_i \in \mathbb{N}\right\}$$
- The filtration
$$V_k = \bigoplus_{\mu \in \Lambda^{(k)}(V)} V_{\lambda+\mu}$$
where
$$\Lambda_\ell\,(V) = \left\{\mu = m_1\varphi_1 + \cdots m_n\varphi_n \mid m_1 \geq m_2 \geq \cdots \geq m_n \geq 0 \quad , \quad \sum_{i=1}^{n} m_i \leq \ell\right\}$$
is a good filtration of V in the sense of Vogan.

Remark 1.1. Actually, in Kato and Ochiai only the K-module structure of V is needed. For example, one can V to be the regular functions on a K-invariant affine variety for which there exists an open orbit of a Borel subgroup of K : a theorem of Vinberg and Kimel'fel'd ([KV]) then says that V is multiplicity free. (Such multiplicity free representations have been classified by Kac. A common setting where both these interpretations of V coexist is when V is taken to be the ring of regular functions on a K-orbit in $\mathfrak{p}$ and the stabilizer of a general point is contained in a Borel subgroup of K (and the orbit corresponds to the associated variety of a $(\mathfrak{g}, K)$-module.)

In this situation, via the Weyl dimension formula, one has
$$\dim V_{\lambda+\mu} = \left(\prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle}\right) \times \left(\prod_{\alpha \in \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_k \rangle + \sum_{i=1}^{n} m_i \langle \alpha, \varphi_i \rangle}{\langle \alpha, \rho_K \rangle}\right)$$

where Δ_K^+ is the set of positive $\mathfrak{k}$ -roots and

$$\Delta_M^+ = \left\{ \alpha \in \Delta_K^+ \mid \langle \alpha, \varphi_i \rangle \neq 0 \quad , \quad i = 1, \ldots, q \right\}$$

Theorem 1.2 (Kato and Ochiai, [KO). *] If ℓ is large, then*

$$\dim V_\ell = c\ell^d d! + (\textit{lower order terms})$$

where

$$d = n + \left| \Delta_M^+ \right|$$

and

$$c = d!n! \left(\prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \right) \int_{\mathcal{S}_n} \prod_{\alpha \in \Delta_M^+} \frac{\sum_{i=1}^n x_i \langle \alpha, \varphi_i \rangle}{\langle \alpha, \rho_K \rangle} d^n x$$

the domain of integration being

$$\mathcal{S}_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq \ x_2 \ \geq \cdots \geq \ x_n \geq 0 \quad , \quad \sum_{i=1}^n x_i \leq 1 \right\}$$

Of course, the constants d and c are also known as, respectively, the Gelfand-Kirillov dimension and the Bernstein degree of the $(\mathfrak{g}, K)$-module V.

Kato and Ochiai then procede to evaluate the integral

$$\int_{\mathcal{S}_n} \prod_{\alpha \in \Delta_M^+} \frac{\sum_{i=1}^n x_i \langle \alpha, \varphi_i \rangle}{\langle \alpha, \rho_K \rangle} d^n x \tag{1}$$

for the case when V is the $(\mathfrak{g}, K)$-module is an irreducible unitary highest weight module of scalar type whose K-types can be identified with the K-types of the ring of regular functions on certain K-orbits in $\mathfrak{p}^+$. In their situation, the roots $\alpha \in \Delta_M^+$ break up into two subsets

$$\begin{aligned} \Sigma_{long} &= \left\{ \alpha \in \Delta_M^+ \mid \alpha = \frac{e_i - e_j}{2} \mid \text{for some } 1 \leq i < j \leq 1 \right\} \\ \Sigma_{short} &= \left\{ \alpha \in \Delta_M^+ \mid \alpha = \frac{e_i}{2} \mid \text{for some } 1 \leq i \leq n \right\} \end{aligned}$$

with the roots in Σ_{long} occuring with a common multiplicity s and the roots in Σ_{short} occuring with a common multiplicity r, allowing us to write the integral (1) in the following form

$$J_{n,r,s} = \int_{\mathcal{S}_n} \prod_{i=1}^n (x_i)^r \prod_{1 \leq i < j \leq n} (x_i - x_j)^s \, d^n x$$

which in turn can be evaluated by applying a change of variables to the original Selberg integral (just to convert the domain from $[0,1]^n$ to $\mathcal{S}_n$)..

The case we were studying last year was the case where V is the $(\mathfrak{g}, K)$-module of a certain family of small unitary representations associated with simple non-Euclidean Jordan algebras. There we also had a multiplicity free $(\mathfrak{g}, K)$-module and a splitting of the roots in Δ_M^+ as

$$\begin{aligned} \Sigma_{short} &= \left\{ \alpha \in \Delta_M^+ \mid \alpha = \frac{e_i \pm e_j}{2} \mid \text{for some } 1 \leq i < j \leq n \right\} \\ \Sigma_{long} &= \left\{ \alpha \in \Delta_M^+ \mid \alpha = e_i \mid \text{for some } 1 \leq i \leq n \right\} \end{aligned}$$

with again common multiplicities d and p for the roots in Σ_1 and Σ_2. This lead us to an integrals of the form

$$I_{n,d,p} = \int_{\mathcal{S}_n} \prod_{i=1}^n (x_i)^p \prod_{1 \leq i < j \leq n} \left(x_i^2 - x_j^2 \right)^d d^n x$$

Unfortunately, having a product of difference of squares of the variables as opposed simply a product of differences of the variables makes this integral much difficult to compute (or at least previously uncomputed).

In this seminar I'll present several different ways of dealing with this integral (each with its own advantages and disadvantages).

We also note that the appearance of the factors $x_i - x_j$ rather than $x_i^2 - x_j^2$ is reflective of the fact that in the Nashiyama and Ochiai's situation the restricted root systems are of type A_{n-1}, while in our situation the restricted roots systems are of type C_n or D_n. Surmounting the consequences of this seemingly minor modification is the crux of the problem addressed in this seminar.

2. Explicit Evaluation of the Integral for $d \in 2\mathbb{Z}_+$

Let

$$I_{n,d,p} = \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i\right)^p \left(\prod_{1\le i<j\le n} \left(x_i^2 - x_j^2\right)\right)^d d^n x$$

where

$$\mathcal{S}_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0 \quad , \quad \sum_{i=1}^n x_i \le 1 \right\}$$

in this section we shall assume that d is a positive even integer.

When d is a even integer the integrand is a symmetric polynomial and we can reformulate the integral as an integral over a much simpler domain:

$$I_{n,d,p} = \frac{1}{n!} \int_{\Omega_n} \left(\prod_{i=1}^n x_i\right)^p \left(\prod_{1\le i<j\le n} \left(x_i^2 - x_j^2\right)\right)^d d^n x$$

where

$$\Omega_n = \mathfrak{S}_n \cdot \mathcal{S}_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge 0, \ \cdots, x_n \ge 0 \quad , \quad \sum_{i=1}^n x_i \le 1 \right\}$$

where Ω_n is the much simpler simplex

$$\Omega_n = \mathfrak{S}_n \cdot \mathcal{S}_n = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_i \ge 0 \ , \ i = 1, \ldots n \ , \quad \text{and} \quad \sum_{i=1}^n x_i \le 1 \right\}$$

Next, we set

$$a = n\left(p + d\left(n-1\right) + 1\right)$$

and note that degree of the integrand is $a - n$.

We now use the identity

$$\Gamma\left(a+1\right) \int_{\Omega_n} f\left(\mathbf{x}\right) d\mathbf{x} = \int_{\Omega_n \times (0,\infty)} f\left(\mathbf{x}\right) s^a e^{-s} d\mathbf{x} ds$$

and make a change of variables

$$\begin{array}{c} y_i = s x_i \\ t = s\left(1 - \sum_{i=1}^n x_i\right) \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} x_i = \frac{y_i}{t + \sum_{k=1}^n y_k} \\ s = t + \sum_{k=1}^n y_k \end{array}$$

which maps $\Omega_n \times (0, \infty)$ diffeomorphically onto $(0,\infty)^n \times (0,\infty)$. The Jacobian this transformation is easily seen to be

$$\frac{\partial\left(y,t\right)}{\partial\left(x,s\right)} = s^n = \left(t + \sum_{k=1}^n y_k\right)^n$$

and so we have for any function f homogeneous of degree $a-n$

$$\begin{aligned}\Gamma\left((a+1)\right)\int_{\Omega_n} f\left(\mathbf{x}\right)d\mathbf{x} &= \int_{\Omega_n\times(0,\infty)} f\left(\mathbf{x}\right)s^a e^{-s}d\mathbf{x}ds \\ &= \int_{(0,\infty)^n\times(0,\infty)} f\left(\frac{\mathbf{y}}{t+\sum_{k=1}^n y_k}\right)\left(t+\sum_{k=1}^n y_k\right)^a e^{-t-\sum_{k=1}^n y_k}\left(t+\sum_{k=1}^n y_k\right)^n d\mathbf{y}dt \\ &= \int_{(0,\infty)^n\times(0,\infty)} f\left(\mathbf{y}\right)e^{-t-\sum_{k=1}^n y_k}d\mathbf{y}dt \\ &= \int_{(0,\infty)^n} f\left(\mathbf{y}\right)e^{-\sum_{k=1}^n y_k}d\mathbf{y}\end{aligned}$$

We thus arrive at

$$I_{n,d,p} = \frac{1}{n!}\frac{1}{\Gamma\left(a+1\right)}\int_{(0,\infty)^n}\left(\prod_{i=1}^n y_i^p\right)\left(\prod_{1\le i<j\le n}\left(y_i^2-y_j^2\right)^d\right)e^{-\sum_{k=1}^n y_k}d\mathbf{y}$$

The next step is to expand the second product using the identity

$$\prod_{1\le i<j\le n}\left(y_i^2-y_j^2\right) = \det\left(y_j^{2(i-1)}\right)_{\substack{i=1,\dots,n\\ j=1,\dots,n}} = \sum_{\sigma\in\mathfrak{S}_n} sgn\left(\sigma\right)\prod_{i=1}^n y_i^{2(\sigma(i)-1)}$$

We have

$$\begin{aligned}\left(\sum_{\sigma\in\mathbb{S}_n} sgn\left(\sigma\right)\prod_{i=1}^n y_i^{2(\sigma(i)-1)}\right)^d &= \sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n} sgn\left(\sigma_1\right)\cdots sgn\left(\sigma_d\right)\prod_{i=1}^n y_i^{2(\sigma_1(i)-1)+\cdots+2(\sigma_d(i)-1)} \\ &= \sum_{\sigma\in\mathfrak{S}_n}\cdots\sum_{\sigma_{dd}\in\mathfrak{S}_n} sgn\left(\sigma_1\cdots\sigma_d\right)\prod_{i=1}^n y_i^{2\left(\sum_{j=1}^d\sigma_j(i)\right)-2d}\end{aligned}$$

and so

$$\begin{aligned}I_{n,d,p} &= \frac{1}{n!}\frac{1}{\Gamma\left(a+1\right)}\sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n} sgn\left(\sigma_1\cdots\sigma_d\right)\int_{(0,\infty)^n}\prod_{i=1}^n y_i^{2\left(\sum_{j=1}^d\sigma_j(i)\right)-2d+p}e^{-\sum_{k=1}^n y_k}d\mathbf{y} \\ &= \frac{1}{n!}\frac{1}{\Gamma\left(a+1\right)}\sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n} sgn\left(\sigma_1\cdots\sigma_d\right)\prod_{i=1}^n\left(\int_0^\infty y_i^{2\left(\sum_{j=1}^d\sigma_j(i)\right)-2d+p}e^{-y_i}dy_i\right) \\ &= \frac{1}{n!}\frac{1}{\Gamma\left(a+1\right)}\sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n} sgn\left(\sigma_1\cdots\sigma_d\right)\prod_{i=1}^n\Gamma\left(2\sigma_1\left(i\right)+\cdots+2\sigma_d\left(i\right)-2d+p+1\right)\end{aligned}$$

where we have used the Euler's integral formula for the gamma function

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1}e^{-t}dt \quad .$$

In summary,

Proposition 2.1. *If d is even and*

$$I_{n,d,p} \equiv \int_{\mathcal{S}_n}\left(\prod_i x_i^p\right)\left(\prod_{1\le i<j\le n}\left(x_i^2-x_j^2\right)^d\right)d^n x$$

then

$$I_{n,d,p}=\frac{1}{n!}\frac{1}{\Gamma\left(a+1\right)}\sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n}sgn\left(\sigma_1\cdots\sigma_d\right)\prod_{i=1}^{n}\Gamma\left(2\sigma_1\left(i\right)+\cdots+2\sigma_d\left(i\right)-2d+p+1\right)$$

where $a=n\left(p+d\left(n-1\right)+1\right)$.

We now focus our attention on the product of gamma factors

$$\prod_{i=1}^{n}\Gamma\left(2\sigma_1\left(i\right)+\cdots+2\sigma_d\left(i\right)-2d+p+1\right)$$

We begin by forming the vector

$$\left[\sum_{j=1}^{d}\sigma_j\left(1\right),\sum_{j=1}^{d}\sigma_j\left(2\right),\ldots,\sum_{j=1}^{d}\sigma_j\left(n\right)\right]$$

and then reordering the components in increasing order to form a vector

$$\boldsymbol{\gamma}\left(\boldsymbol{\sigma}\right)\equiv\left[\min_i\left\{\sum_{j=1}^{d}\sigma_j\left(i\right)\right\},\ldots,\max_i\left\{\sum_{j=1}^{d}\sigma_j\left(i\right)\right\}\right]$$

Lemma 2.2. *For any arrangement* $\boldsymbol{\sigma}\in\left(\mathfrak{S}_n\right)^d$ *we have*

$$\gamma_i\left(\boldsymbol{\sigma}\right)\geq\frac{d}{2}\left(i+1\right)$$

Proof. We first note that

$$\gamma_1\left(\boldsymbol{\sigma}\right)\geq d$$

follows readily from the requirement that each $\sigma_j\left(i\right)\geq 1$. Next we note that for any arrangement $\left(\sigma_1,\ldots,\sigma_d\right)$

$$\sum_{j=1}^{d}\sum_{i=1}^{n}\sigma_i\left(j\right)=\sum_{j=1}^{d}\left(1+2+\cdots+n\right)=\frac{d}{2}n\left(n+1\right)$$

and that the particular arrangement where

$$\sigma_k=\begin{cases}\left[1,2,\ldots,n-1,n\right] & \text{if } k \text{ is even}\\ \left[n,n-1,\ldots,2,1\right] & \text{if } k \text{ is odd}\end{cases}$$

leads to

$$\begin{aligned}\boldsymbol{\gamma}\left(\boldsymbol{\sigma}\right)&=\left[\frac{d}{2}\left(1\right)+\frac{d}{2}\left(n\right),\frac{d}{2}\left(2\right)+\frac{d}{2}\left(n-1\right),\ldots,\frac{d}{2}\left(n\right)+\frac{d}{2}\left(1\right)\right]\\&=\left[\frac{d}{2}\left(n+1\right),\frac{d}{2}\left(n+1\right),\ldots,\frac{d}{2}\left(n+1\right)\right]\end{aligned}$$

Now note that one cannot decrease the last component of $\boldsymbol{\gamma}\left(\boldsymbol{\sigma}\right)$ further without violating the requirement that $\sum_{i=1}^{n}\gamma_i\left(\boldsymbol{\sigma}\right)=\frac{d}{2}n\left(n-1\right)$ (and the stipulated ordering $\gamma_i\left(\boldsymbol{\sigma}\right)\leq\gamma_{i+1}\left(\boldsymbol{\sigma}\right)$). Thus, we have

$$\gamma_n\left(\boldsymbol{\sigma}\right)\geq\frac{d}{2}\left(n+1\right)$$

Finally, we observe that the arrangement

$$\sigma_k=\begin{cases}\left[1,2,\ldots,i-1,i,i+1,\ldots,n\right] & \text{if } k \text{ is even}\\ \left[i,i-1,\ldots,2,1,i+1,\ldots,n\right] & \text{if } k \text{ is odd}\end{cases}$$

leads to

$$\boldsymbol{\gamma}\left(\boldsymbol{\sigma}\right)=\left[\frac{d}{2}\left(i+1\right),\ldots,\frac{d}{2}\left(i+1\right),d\left(i+1\right),\ldots,d\left(n\right)\right]$$

and that by essentially the reasoning as above

$$\gamma_i(\boldsymbol{\sigma}) \geq \frac{d}{2}(i+1)$$

∎

Corollary 2.3. *Let*

$$S_{n,d}(p) = \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1)$$

Then

$$S_{n,d}(p) = \Phi_{n,d}(p) \prod_{i=1}^{n} \Gamma(p + 1 + d(i-1))$$

where

$$\Phi_{n,d}(p) = \left(\sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} (p + 1 + d(i+1))_{2\mu_i(\sigma)} \right)$$

is a polynomial in p of degree $\leq \frac{d}{2} n(n-1)$.

Proof. Set

$$\mu_i(\boldsymbol{\sigma}) = \gamma_i(\boldsymbol{\sigma}) - \frac{d}{2}(i+1)$$

so that

$$\mu_i(\boldsymbol{\sigma}) \geq 0 \qquad , \qquad i = 1, \ldots, n$$

For each term of the iterated sum we can arrange the gamma factors so that their arguments are non-decreasing. In other words we can write

$$\begin{aligned} S_{n,d}(p) &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} \Gamma(2\gamma_i(\boldsymbol{\sigma}) - 2d + p + 1) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} \Gamma(2\mu_i(\boldsymbol{\sigma}) + d(i+1) - 2d + p + 1) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} \Gamma(p + 1 + d(i-1) + 2\mu_i(\boldsymbol{\sigma})) \end{aligned}$$

We now introducing the Pochhammer symbols $(x)_n$ defined by

$$(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}$$

and note that for positive integers n

$$(x)_n = (x)(x+1) \cdots (x+n-1)$$

and

$$(x)_0 = 1.$$

Returning to our expression for $S_{n,d}(p)$ we have

$$\begin{aligned} S_{n,d}(p) &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} \Gamma(p + 1 + 2\mu_i(\sigma) + d(i-1)) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} (p + 1 + d(i-1))_{2\mu_i(\sigma)} \Gamma(p + 1 + d(i-1)) \\ &= \left(\sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\rho_d \in \mathfrak{S}_n} sgn(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} (p + 1 + d(i-1))_{2\mu_i(\sigma)} \right) \prod_{i=1}^{n} \Gamma(p + 1 + d(i+1)) \end{aligned}$$

Finally we note that each factor

$$\prod_{i=1}^{n}\left(d\left(i-1\right)+p+1\right)_{2\mu_i(\sigma)}$$

is a polynomial in p of total degree

$$\begin{aligned}\sum_{i=1}^{n}2\mu_i\left(\sigma\right) &= \sum_{i=1}^{n}\left(2\gamma_i\left(\boldsymbol{\sigma}\right)-d\left(i+1\right)\right)\\ &= 2\sum_{i=1}^{n}\gamma_i\left(\boldsymbol{\sigma}\right)-d\left(\frac{1}{2}n\left(n+1\right)+n\right)\\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_j\left(i\right)-d\left(\frac{1}{2}n\left(n+1\right)+n\right)\\ &= 2d\left(\sum_{i=1}^{n}i\right)-d\left(\frac{1}{2}n\left(n+1\right)+n\right)\\ &= 2d\frac{1}{2}n\left(n+1\right)-\frac{d}{2}n\left(n+1\right)-dn\\ &= \frac{d}{2}n\left(n-1\right)\end{aligned}$$

We conclude that

$$S_{n,d}\left(p\right)=\Phi_{n,d}\left(p\right)\prod_{i=1}^{n}\Gamma\left(p+1+d\left(i-1\right)\right)$$

with

$$\Phi_{n,d}\left(p\right)=\left(\sum_{\sigma\in\mathfrak{S}_n}\cdots\sum_{\rho_d\in\mathfrak{S}_n}sgn\left(\sigma_1\cdots\sigma_d\right)\prod_{i=1}^{n}\left(p+1+d\left(i+1\right)\right)_{2\mu_i(\sigma)}\right)\tag{(2)}$$

a polynomial of degree $\leq\frac{d}{2}n\left(n-1\right)$. ∎

We remark here that the bound $\deg\left(\Phi_{n,d}\left(p\right)\right)\leq\frac{d}{2}n\left(n-1\right)$ is easily seen to be less than optimal: the Pochhamer products

$$\prod_{i=1}^{n}\left(p+1+d\left(i+1\right)\right)_{2\mu_i(\sigma)}$$

are all monic polynomials, and so when we sum over the arrangements in $\left(\mathfrak{S}_n\right)^d$, the $sgn\left(\sigma_1\cdots\sigma_d\right)$ factors will lead to a complete cancellation of the leading terms. In fact, explicit computations of the right hand side of (2) reveal that at least for small n and d the actual degree of $\Phi_{n,d}\left(p\right)$ is $\frac{d}{4}n\left(n-1\right)$; that is, that that the terms of degree $\frac{d}{2}n\left(n-1\right),\frac{d}{2}n\left(n-1\right)-1,\ldots,\frac{d}{4}n\left(n-1\right)+1$ all, quite remarkably, cancel. Unfortunately, we have yet to find a direct combinatorial argument as to why the first $\frac{d}{4}n\left(n-1\right)$ leading terms all cancel. However, in §3 we shall succeed not only in extending our results to the case of odd d, but we will also obtain a least upper bound on the the degree of the polynomial factor $\Phi_{n,d}\left(p\right)$ for arbitrary positive integers d and n.

3. Evaluation of the Integral for $d\in\mathbb{Z}_+$

3.1. **Symmetric polynomials and an integral formula of Macdonald.** Most of the following material is very classical and well known; we include it simply to keep the exposition of our notation self-contained.

By a symmetric polynomial of n variables we mean a polynomial $p\in\mathbb{C}\left[x_1,\ldots,x_n\right]$ invariant under the natural of the symmetric group $\mathfrak{S}_n$:

$$p\left(x_1,\ldots,x_n\right)=\left(\sigma\cdot p\right)\left(x_1,\ldots,x_n\right)\equiv p\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)\qquad\forall\ \sigma\in\mathfrak{S}_n\quad.$$

The action of $\mathfrak{S}_n$ preserves the subspaces of $\mathbb{C}[x_1,\ldots,x_n]$ consisting of homogeneous polynomials of fixed total degree. We shall denote by $\Lambda^m_{(n)}$ the space of homogeneous symmetric polynomials in n variables of total degree m, so that

$$\Lambda_{(n)} \equiv \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_n} = \bigoplus_{m=0}^{\infty} \Lambda^m_{(n)}$$

There are several fundamental bases for the subspaces $\Lambda^m_{(n)}$, each parameterized by partitions of length n and weight m. A partition λ of length n is simply a non-increasing list of non-negative integers; i.e., $\lambda = [\lambda_1,\lambda_2,\ldots,\lambda_n]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The weight $|\lambda|$ of λ is the sum of its parts; i.e., $|\lambda| = \sum_{i=1}^n \lambda_i$. There is a partial ordering of the set of partitions of weight w, called the *dominance partial ordering*, defined as follows

$$\mu \leq \lambda \quad \Longrightarrow \quad \sum_{j=1}^{i} \left(\lambda_j - \mu_j\right) \geq 0 \qquad , \qquad \text{for } i = 1,2,\ldots,n$$

Let $\lambda = [\lambda_1,\ldots,\lambda_n]$ be a partition, and let $x^\lambda = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ be the corresponding monomial. The *monomial symmetric function* m_λ is the sum of all distinct monic monomials that can be obtained from x^λ by permuting the x_i's. Every homogeneous symmetric polynomial of degree m can be uniquely expressed as a linear combination of the m_λ with $|\lambda| = m$.

The *power sum* symmetric polynomials are defined as follows. For each $r \geq 1$, let

$$p_r = m_{(r)} = \sum_{i=1}^{n} x_i^r$$

and then for any partition λ, set

$$p_\lambda(x) = p_{\lambda_1}(x)\, p_{\lambda_2}(x) \cdots p_{\lambda_n}(x)$$

The power sum symmetric functions $p_\lambda(x)$ with $|\lambda| = m$ provide another basis for the homogeneous symmetric polynomials of degree m. In what follows, the power sum symmetric functions are only used to define a particular inner product for the symmetric polynomials; i.e., an inner product will be defined by specifying matrix entries with respect to the basis of power sum symmetric polynomials.

The *Schur polynomials* provide yet another basis. These can be defined as follows. For any partition λ of length n,

$$a_\lambda(x) = \det\left(x_i^{\lambda_j}\right)$$

The $a_\lambda(x)$ are obviously odd with respect to the action of the permutation group $\mathfrak{S}_n$; i.e. $a_\lambda(\sigma(x)) = sgn(\sigma)\, a_\lambda(x)$ for all $\sigma \in \mathfrak{S}_n$. However, it turns out that $a_\lambda = 0$ for and $\lambda < \delta \equiv [n-1,n-2,\ldots,1,0]$, and that, for every partition λ, $a_\delta(x)$ divides $a_{\lambda+\delta}(x)$. Indeed,

$$s_\lambda(x) \equiv \frac{a_{\lambda+\delta}(x)}{a_\delta(x)}$$

is a symmetric polynomial of degree $|\lambda|$. The polynomials $s_\lambda(x)$ are the Schur symmetric polynomials. The Schur polynomials $\{s_\lambda \mid |\lambda| = m\}$ provide another fundamental basis for the symmetric polynomials that are homogeneous of degree m. A special case that will be important to us later on is

$$s_\delta(x) = \prod_{1\leq i<j\leq n} (x_i + x_j)$$

where again $\delta = [n-1,n-2,\ldots,1,0]$. We note also the fact that $a_\delta(x)$ is just the Vandermonde determinant

$$a_\delta(x) == \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}^{n-1} & x_{n-1}^{n-2} & \cdots & x_{n-1} & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} = \prod_{1\leq i<j\leq n} (x_i - x_j) \qquad .$$

Jack's symmetric functions $P_\lambda^{(\alpha)}$ are symmetric functions indexed by partitions and depending rationally on a parameter α which interpolate between the Schur functions $s_\lambda(x)$, the monomial symmetric functions and two other bases associated with spherical symmetric polynomials on symmetric spaces. other $Z_\lambda(x)$. They are (uniquely) characterized by two properties

(i) $P_\lambda^{(\alpha)}(x) = m_\lambda(x) + \sum_{\substack{\mu<\lambda \\ |\mu|=|\lambda|}} c_{\lambda,\mu} m_\mu(x)$; that is, the "leading term" of $P_\lambda^{(\alpha)}$ is the monomial symmetric functions m_λ and the remaining terms involve only monomial symmetric functions m_μ for which the partition index μ is less than λ with respect to the dominance ordering.

(ii) When one defines a scalar product on the vector space of homogeneous polynomials of degree m by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} \prod_{r=1}^{n} \left(r^{m_\lambda(r)} \cdot m_\lambda(r)! \right)$$

where $m_\lambda(r)$ is the number of times the integer r appears in λ, then

$$\left\langle P_\lambda^{(a)}, P_\mu^{(\alpha)} \right\rangle = 0 \qquad \text{if } \mu \neq \lambda \quad .$$

We note that this definition basically ensures that the $P_\lambda^{(\alpha)}(x)$ are constructible via a Gram-Schmidt process (although not quite straightforwardly, as the dominance ordering is only a partial ordering).

In fact, the Jack symmetric polynomials $P_\lambda^{(\alpha)}$ for special values $\alpha = 2,\ 1,$ and $\frac{1}{2}$ can also be characterized as the spherical polynomials for, respectively, $GL(n,\mathbb{R})/O(n)$, $GL(n,\mathbb{H})/U(\mathbb{H})$. When $\alpha = 1$ the Jack symmetric polynomials coincide with the spherical polynomials for $GL(n,\mathbb{C})/U(n)$, as well the Schur polynomials.

Next we recall the following well known property of Schur polynomials

$$s_\mu(x) s_\nu(x) = \sum_{\substack{\lambda \leq \mu+\nu \\ |\lambda|=|\mu|+|\nu|}} K_{\mu\nu}^\lambda s_\lambda$$

where the coefficients $K_{\mu\nu}^\lambda$ are determined by the Littlewood-Richardson rule (and are in fact interpretable as the Clehsch-Gordon coefficents for $SL(n)$). Because of the triangular decomposition of the product of two Schur polynomials in terms of other Schur polynomials, and the triangular decomposition of a Jack symmetric polynomial (and Schur polynomials in particular) in terms of the monomial symmetric functions we can infer that

$$(s_\delta(x))^d = \sum_{\substack{\lambda \leq d\delta \\ |\lambda| \leq d|\delta|}} c_\lambda^{(\alpha)} P_\lambda^{(\alpha)}(x) \tag{3}$$

for suitable coefficients $c_\lambda^{(\alpha)}$ (To be a little more forthright, we simply note that

$$span\left\{ s_\lambda \mid |\lambda| = |\delta| \ ,\ \lambda \leq \delta \right\} = span\left(m_\lambda \mid |\lambda| = |\delta| \ ,\ \lambda \leq \delta \right) = span\left\{ P_\lambda^{(\alpha)} \mid |\lambda| = |\delta| \ ,\ \lambda \leq \delta \right\}$$

which follows immediately from condition (i) in the definition of the $P_\lambda^{(\alpha)}$ and the linear independence of the $P_\lambda^{(\alpha)}$ which follows immediately from orthogonality property (ii) in the definition of the $P_\lambda^{(\alpha)}$.)

Next we quote a specialization of a result of Macdonald that is derived by applying a particular change of variables applied to a formula due to Gross and Richards [GR], and Kadell [K].

Proposition 3.1 ([Mac2]. *, pg. 386]For* $Re(k) > 0$*,* $Re(r) > 0$*,*

$$\int_{\mathcal{S}_n} P_\lambda^{\left(\frac{1}{k}\right)}(x) \prod_{i=1}^{n} (x_i)^{r-1} (a_\delta(x))^{2k} dx = \frac{1}{\Gamma(a+1)} v_\lambda(k) \prod_{i=1}^{n} \Gamma(\lambda_i + r + k(n-i))$$

where

$$a = |\lambda| + rn + n(n-1)k$$
$$v_\lambda(k) = \prod_{1\le i<j\le n} \frac{\Gamma(\lambda_i - \lambda_j + k(j-i+1))}{\Gamma(\lambda_i - \lambda_j + k(j-i))}$$

We'll now derive a formula for

$$I_{n,d,r} = \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^{r-1}\right)\left(\prod_{1\le i<j\le n}\left(x_i^2 - x_j^2\right)\right)^d dx$$

We begin by noting that the second factor in the integrand can be written as

$$\left(\prod_{1\le i<j\le n}\left(x_i^2 - x_j^2\right)\right)^d = \left(\prod_{1\le i<j\le n}(x_i + x_j)\right)^d \left(\prod_{1\le i<j\le n}(x_i - x_j)\right)^d = (s_\delta(x))^d (a_\delta(x))^d$$

And so

$$I_{n,1,r} = \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^{r-1}\right)(s_\delta(x))^d (a_\delta(x))^d dx$$

We now employ (3) with $\alpha = \frac{d}{2}$ we get

$$\begin{aligned}
I_{n,d,r} &= \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^{r-1}\right)\left(\sum_{\substack{\lambda\le d\delta\\ |\lambda|=d|\delta|}} c_\lambda^{\left(\frac{2}{d}\right)} P_\lambda^{\left(\frac{2}{d}\right)}(x)\right)(a_\delta(x))^d dx\\
&= \sum_{\substack{\lambda\le d\delta\\ |\lambda|=d|\delta|}} c_\lambda^{\left(\frac{2}{d}\right)} \int_{\mathcal{S}_n}\left(\prod_{i=1}^n x_i^{r-1}\right) P_\lambda^{\left(\frac{2}{d}\right)}(x)(a_\delta(x))^d dx\\
&= \frac{1}{\Gamma(a+1)} \sum_{\substack{\lambda\le d\delta\\ |\lambda|=d|\delta|}} c_\lambda^{\left(\frac{2}{d}\right)} v_\lambda\left(\frac{d}{2}\right)\prod_{i=1}^n \Gamma\left(\lambda_i + r + \frac{d}{2}(n-i)\right)\\
&= \sum_{\substack{\lambda\le d\delta\\ |\lambda|=d|\delta|}} c_\lambda \prod_{i=1}^n \Gamma\left(\lambda_i + r + \frac{d}{2}(n-i)\right)
\end{aligned}$$

for suitable coefficients c_λ which depend only on n and d.

Lemma 3.2. *If $\lambda = [\lambda_1, \ldots, \lambda_n]$ is a partition of weight $d|\delta|$ such that $\lambda \le d\delta$, then*

$$\lambda_i \ge \mu_i \equiv \left[\frac{d(n-i)+1}{2}\right]$$

Proof. Let $\mathcal{P}$ be the set of partitions $\lambda = [\lambda_1, \ldots, \lambda_n]$ satisfying the criteria

$$\begin{aligned}
&\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\\
&\sum_{i=1}^n \lambda_i = \frac{d}{2} n(n-1)\\
&\sum_{j=1}^i \lambda_j \le \sum_{j=1}^i d(n-j) = ni - \frac{1}{2} i(i+1) = \frac{d}{2} i(2n-i-1)
\end{aligned}$$

Since the total weight of such a λ is fixed, in order to minimize a particular part λ_i we need to arrange it so that the parts λ_j to the left of λ_i are as large as possible while the λ_j to the right of λ_i are also large as possible (otherwise we could shift some of λ_i's weight to the right). In fact, if

$$\mu_i \equiv \min_{\lambda\in\mathcal{P}} \lambda_i$$

we'll need

$$\mu_i = \mu_{i+1} = \cdots = \mu_n$$

to make the μ_j to the right as large as possible and

$$\mu_j = d\,(n-j) \qquad , \qquad j = 1,\ldots,i-1$$

for the μ_j to the left to be as large as possible. But for such a minimizing configuration μ, we must also have

$$\begin{aligned}\frac{d}{2}n\,(n-1) &= \sum_{j=1}^{n}\mu_j = \sum_{j=1}^{i-1} d\,(n-j) + (n-i+1)\,\mu_i \\ &= dn\,(i-1) - \frac{d}{2}i\,(i-1) + (n-i+1)\,\mu_i\end{aligned}$$

Solving this for μ_i yields

$$\mu_i = \frac{d}{2}\,(n-i)$$

However, if d is odd then

$$\mu_i = \frac{d}{2}\,(n-i)$$

will be an integer only when $n-i$ is even. In such a case, the first integer larger than μ_i would be

$$\frac{d}{2}\,(n-i) + \frac{1}{2}$$

Accounting for this circumstance, we can write

$$\mu_i = \min_{\lambda\in\mathcal{S}} \lambda_i = \left[\frac{d\,(n-i)+1}{2}\right] \qquad .$$

noting that the added $\frac{1}{2}$ is innocuous in the cases when d is even or when d is odd and $n-i$ is even. ∎

Theorem 3.3. *For $d \in \mathbb{Z}_{>0}$,*

$$I_{n,d,r-1} = \Phi\,(r)\prod_{i=1}^{n}\Gamma\left(r + \mu_i + \frac{d}{2}\,(n-i)\right)$$

where $\Phi\,(r)$ is a polynomial of degree $\leq \frac{d}{4}n\,(n-1)$ if d is even, or of degree $\leq \frac{d}{4}n\,(n-1) - \frac{1}{2}\left[\frac{n}{2}\right]$ if d is odd..

Proof. We have

$$\begin{aligned}I_{n,d}\,(r) &= \frac{1}{\Gamma\,(a+1)}\sum_{\substack{\lambda\leq d\delta\\|\lambda|=d|\delta|}} C_\lambda \prod_{i=1}^{n}\Gamma\left(\lambda_i - \mu_i + \mu_i + r + \frac{d}{2}\,(n-i)\right) \\ &= \frac{1}{\Gamma\,(a+1)}\sum_{\substack{\lambda\leq d\delta\\|\lambda|=d|\delta|}} C_\lambda \prod_{i=1}^{n}\left(\mu_i + r + \frac{d}{2}\,(n-i)\right)_{\lambda_i-\mu_i}\Gamma\left(\mu_i + r + \frac{d}{2}\,(n-i)\right) \\ &= \frac{1}{\Gamma\,(a+1)}\left(\sum_{\substack{\lambda\leq d\delta\\|\lambda|=d|\delta|}} C_\lambda \prod_{i=1}^{n}\left(\mu_i + r + \frac{d}{2}\,(n-i)\right)_{\lambda_i-\mu_i}\right)\prod_{i=1}^{n}\Gamma\,(r + d\,(n-i))\end{aligned}$$

The upper bound on the total degree of the product of Pochhammer symbols with the large parentheses is

$$\begin{aligned}\sum_{\iota=1}^{n}(\lambda_i-\mu_i) &= \sum_{i=1}^{n}\lambda_i-\sum_{i=1}^{n}\mu_i\\ &= \frac{d}{2}n(n-1)-\sum_{i=1}^{n}\left[\frac{d(n-i)+1}{2}\right]\end{aligned}$$

Now when d is even, we have $\left[\frac{d(n-i)+1}{2}\right]=\frac{d}{2}(n-i)$ and so

$$\begin{aligned}\sum_{\iota=1}^{n}(\lambda_i-\mu_i) &= \frac{d}{2}n(n-1)-\sum_{i=1}^{n}\frac{d(n-i)}{2}=\frac{d}{2}n(n-1)-\frac{d}{2}n^2+\frac{d}{4}n(n+1)\\ &= \frac{d}{4}n(n-1) \qquad (\text{ if } d\in 2\mathbb{Z})\end{aligned}$$

When d is odd, we have

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{d(n-i)+1}{2}\right] &= \sum_{i=1}^{n}\frac{d}{2}(n-i)+\sum_{\substack{(n-i)\\ odd}}\frac{1}{2}\\ &= \frac{d}{4}n(n-1)+\sum_{\substack{(n-i)\\ odd}}\frac{1}{2}\end{aligned}$$

Now if $n=2k$, then $(n-i)$ will be odd exactly k times as i ranges from 1 to n, and if $n=2k+1$, $n-i$ will be odd exactly k times. which is the stated bound. We can thus write

$$\sum_{i=1}^{n}\left[\frac{d(n-i)+1}{2}\right]=\frac{d}{4}n(n-1)+\frac{1}{2}\left[\frac{n}{2}\right]$$

and so

$$\sum_{\iota=1}^{n}(\lambda_i-\mu_i)=\frac{d}{4}n(n-1)-\frac{1}{2}\left[\frac{n}{2}\right] \qquad , \qquad d\in[1]_2$$

∎

Remark 3.4. When d is even

$$\mu_i=\frac{d(n-i)}{2}$$

and so

$$\Phi(r)\prod_{i=1}^{n}\Gamma\left(r+\mu_i+\frac{d}{2}(n-i)\right)=\Phi(r)\prod_{i=1}^{n}\Gamma(r+d(n-i))$$

We thus have an indirect proof that

$$\sum_{\sigma_1\in\mathfrak{S}_n}\cdots\sum_{\sigma_d\in\mathfrak{S}_n}sgn(\sigma_1\cdots\sigma_d)\prod_{i=1}^{n}\Gamma(r+2\sigma_1(i)+\cdots 2\sigma_d(i))=\Phi(r)\prod_{i=1}^{n}\Gamma(r+d(n-i))$$

where $\Phi(r)$ is a polynomial of degree $\frac{d}{4}n(n-1)$.

Remark 3.5. Explicit calculations reveal that for $d=1,2,4$ and $n=2,3,4,5$ that the upper bound on the degree of $\Phi(r)$ is in fact realized. Thus, $\frac{d}{4}n(n-1)$ is, in fact, the least upper bound for general d and n.

4. A Recursive Formula for $\int_{\mathcal{S}_n} e_\lambda(x)\,(a_\delta(x))^d\,d^n x$

Recall that the i^{th} elementary symmetric polynomial in n-variables is

$$e_i^{(n)} = \begin{cases} 0 & \text{if} \quad i < 0 \\ 1 & \text{if} \quad i = 0 \\ \sum_{1 \le j_1 < j_2 < \cdots < j_i \le n} x_{j_1} x_{j_2} \cdots x_{j_i} & \text{if} \quad 1 \le i \le n \\ 0 & \text{if} \quad i > n \end{cases} \tag{1}$$

Alternatively, the $e_i^{(n)}$ can be defined as the coefficient of t^i in

$$E(t) = \prod_{i=1}^{n} (1 + t x_i) \tag{2}$$

Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ be a partition of length n, the *elementary symmetric polynomial* in n variables corresponding to the partition λ is

$$e_\lambda^{(n)} = \prod_{i=1}^{n} e_{\lambda_i}^{(n)} \tag{3}$$

Lemma 4.1. *Let $e_i^{(n-1)}(x)$ be the i^{th} elementary symmetric polynomial in $n-1$ variables $x_1, \ldots, x_{n-1}$. Then*

$$e_i^{(n)}(x) = e_i^{(n-1)}(x) + x_n e_{i-1}^{(n-1)}(x) \quad . \tag{4}$$

This is fairly obvious from the definition (1).

Lemma 4.2. *Let $x + a = (x_1 + a, x_2 + a, \ldots, x_n + a)$. Then*

$$e_i^{(n)}(x+a) = \sum_{j=0}^{i} \binom{n-j}{i-j} a^{i-j} e_j^{(n)}(x)$$

Proof. We start with the generating function $E(t)$ for the elementary symmetric functions,

$$\prod_{i=1}^{n} (1 + t x_i) = \sum_{i=0}^{n} e_i^{(n)}(x)\, t^i \qquad \Longrightarrow \qquad t^n \prod_{i=1}^{n} \left(\frac{1}{t} + x_i \right) = \sum_{i=0}^{n} e_i^{(n)}(x)\, t^i$$

We now replace t by $1/s$ to get

$$s^{-n} \prod_{i=1}^{n} (s + x_i) = \sum_{i=0}^{n} e_i^{(n)}(x)\, s^{-i}$$

which yields an alternative generating function

$$\prod_{i=1}^{n} (s + x_i) = \sum_{i=0}^{n} e_i^{(n)}(x)\, s^{n-i}$$

Let's now replace x_i by $x_i + a$, for $i = 1, \ldots, n$. We then have

$$\begin{aligned} \sum_{i=0}^{n} e_i^{(n)}(x+a)\, s^{n-i} = \prod_{i=1}^{n} (s + x_i + a) &= \sum_{i=0}^{n} e_i^{(n)}(x)\,(s+a)^{n-i} \\ &= \sum_{i=0}^{n} e_i^{(n)}(x) \left(\sum_{j=0}^{n-i} \binom{n-i}{j} a^j s^{n-i-j} \right) \end{aligned}$$

Comparing the total coefficients of s^{n-i} on both sides we see

$$
\begin{aligned}
e_i^{(n)}(x+a) &= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\delta_{n-i,n-k-j}\left(\begin{array}{c} n-k \\ j \end{array}\right) a^j e_k^{(n)}(x) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\delta_{i,k+j}\left(\begin{array}{c} n-k \\ j \end{array}\right) a^j e_k^{(n)}(x) \\
&= \sum_{k=0}^{i}\left(\begin{array}{c} n-k \\ i-k \end{array}\right) a^{i-k} e_k^{(n)}(x) \\
&= \sum_{j=0}^{i}\left(\begin{array}{c} n-j \\ i-j \end{array}\right) a^{i-j} e_j^{(n)}(x)
\end{aligned}
$$

∎

4.1. **Selberg Integrals of Elementary Symmetric Functions.** Let $\mathcal{S}_n \subset \mathbb{R}^n$ be the simplex prescribed by[1]

$$
\mathcal{S}_n = \left\{ x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0 \quad , \quad \sum_{i=1}^{n} x_i \leq 1 \right\}
$$

Let $\Delta^{(n)}(x)$ be the Vandermonde determinant:

$$
\Delta^{(n)}(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)
$$

We aim to derive a recursive formula for integrals of the form

$$
I_{n,\lambda,d} \equiv \int_{\mathcal{S}_n} e_\lambda^{(n)}(x) \left(\Delta^{(n)}(x)\right)^d d^n x
$$

where λ is an arbitary partition of length $\leq n$ and d is a positive integer.

Remark 4.3. The application we have in mind is the following. If $s_\lambda^{(n)}$ is a Schur symmetric polynomial, the Giambelli formula (see page 453 of Fulton and Harris)

$$
s_\lambda^{(n)} = \left| e_{\mu_i+j-i}^{(n)} \right| = \left| \begin{array}{cccc} e_{\mu_1}^{(n)} & e_{\mu_1+1}^{(n)} & \cdots & e_{\mu+l-1}^{(n)} \\ e_{\mu_2-1}^{(n)} & e_{\mu_2}^{(n)} & \cdots & \\ \vdots & & & \\ e_{\mu_l-l+1}^{(n)} & \cdots & & e_{\mu_l}^{(n)} \end{array} \right|
$$

where $\mu = (\mu_1, \ldots, \mu_l)$ is the partition conjugate to λ. Note that the Giambelli formula effectively (and succinctly) expresses any Schur polynomial as a certain sum products of elementary symmetric polynomials; that is to say, the above formula allows us to express any Schur polynomial $s_\lambda^{(n)}$ as a certain linear combination of the symmetric polynomials $e_{\lambda'}^{(n)}$ with $|\lambda| = |\lambda'|$. Let us write this as

$$
s_\lambda^{(n)} = \sum_{\substack{\mu \\ |\mu|=|\lambda|}} c_{\lambda,\mu} e_\mu^{(n)}(x)
$$

[1]Alternatively, $\mathcal{C}_n$ can be thought of as a fundamental domain for the natural action of the symmetric group on the simplex bounded by the hyperplanes $x_1 = 0,\ x_2 = 0,\ \ldots\ , x_n = 0$ and $\sum_{i=1}^{n} x_i = 1$. However, if d is an odd integer, then the integrand is only skew-symmetric with respect to interchanges of variables.

Now note that

$$e_n^{(n-1)}(x) = \prod_{i=1}^{n} x_i$$
$$s_\delta^{(n)}(x) = \prod_{1\le i<j\le n} (x_i + x_j)$$

and so

$$\left(\prod_{i=1}^{n} x_i\right)^p \left(\prod_{1\le i<j\le n} (x_i + x_j)\right) = \left(e_n^{(n)}\right)^p \left(\sum_{\substack{\mu \\ |\mu|=|\lambda|}} c_{\delta,\mu} e_\mu^{(n)}(x)\right)$$
$$= \sum_{|\mu|=|\lambda|} c_{\delta,\mu} e_{\mu+pe_n}^{(n)}(x)$$

Here $\mu + pe_n$ is the partition obtained from μ be increasing the number of times n occurs in μ by p.

Thus, a formula for $I_{n,\lambda,d}$ will also furnish us with a means to compute integrals of the form

$$\begin{aligned}
J_{n,p,d} &\equiv \int_{\mathcal{S}_n} \left(\prod_{i=1}^{n} x_i\right)^p \left(\prod_{1\le i<j\le n} \left(x_i^2 - x_j^2\right)\right) dx \\
&= \int_{\mathcal{S}_n} \left(\prod_{i=1}^{n} x_i\right)^p \left(\prod_{1\le i<j\le n} (x_i + x_j)\right) \left(\prod_{1\le i<j\le n} (x_i - x_j)\right) dx \\
&= \int_{\mathcal{S}_n} \left(\sum_{|\mu|=|\lambda|} c_{\delta,\mu} e_{\mu+pe_n}^{(n)}(x)\right) \Delta^{(n)}(x)\, dx \\
&= \sum_{|\mu|=|\lambda|} c_{\delta,\mu} I_{n,\mu,1}
\end{aligned}$$

4.2. **A Change of Variables Formula.** Recall

$$\mathcal{S}_n = \left\{ x \in \mathbb{R}^n \mid x_1 \ge \cdots \ge x_n \ge 0 \quad , \quad \sum_{i=1}^{n} x_i \le 1 \right\}$$

Set

$$\begin{aligned}
t_i &= \frac{x_i - x_n}{1 - nx_n} \qquad , \qquad x_i = (1 - t_n)\, t_i + \frac{1}{n} t_n \\
t_n &= nx_n \qquad\qquad , \qquad x_n = \frac{1}{n} t_n
\end{aligned}$$

The Jacobian of this transformation is

$$\left(\frac{\partial x}{\partial t}\right) = \det \begin{bmatrix} 1 - t_n & 0 & \cdots & \frac{1}{n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 - t_n & \frac{1}{n} \\ 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix} = \frac{1}{n} (1 - t_n)^{n-1}$$

and the new region of integration is

$$\left\{ t \in \mathbb{R}^n \mid t_1 \ge t_2 \ge \cdots \ge t_{n-1} \ge 0 \quad , \quad \sum_{i=1}^{n-1} t_i \le 1 \quad , \quad 0 \le t_n \le 1 \right\} \approx \mathcal{S}_{n-1} \times [0, 1]$$

We thus have

Lemma 4.4.

$$\int_{\mathcal{S}_n} f(x)\, d^n x = \int_0^1 \frac{1}{n}(1-t_n)^{n-1}\left(\int_{\mathcal{S}_{n-1}} f(x(t))\, d^{n-1}t\right) dt_n$$

where

$$\begin{aligned} x_i(t) &= (1-t_n)\, t_i + \frac{1}{n} t_n \\ x_n(t) &= \frac{1}{n} t_n \end{aligned}$$

Remark 4.5. If $f(x)$ is polynomial, then after making the change of variables and expanding one can pull all the factors of t_n and $(1-t_n)$ out of the integral over $\mathcal{S}_{n-1}$. And then the integrations over t_n will just be beta type integrals:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} = \beta(\alpha,\beta) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Indeed, if one iterates the change of variables formula, then after $n-1$ reductions, one arrives at

$$\begin{aligned} \int_{\mathcal{S}_n} f(x)\, d^n x &= \int_0^1 \frac{1}{n}(1-t_n)^{n-1}\left(\int_0^1 \frac{1}{n-1}(1-t_{n-1})^{n-2}\left(\cdots\left(\int_{\mathcal{S}_1} f(\cdot)\, dt_1\right)\right) dt_{n-1}\right) dt_n \\ &= \frac{1}{n!}\left(\int_0^1 (1-t_n)^{n-1}\left(\int_0^1 (1-t_{n-1})^{n-2}\left(\cdots\left(\int_0^1 f(x(t))\, dt_1\right)\right) dt_{n-1}\right) dt_n\right) \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= (1-t_n)(1-t_{n-1})\cdots(1-t_2)\, t_1 + \frac{1}{2}(1-t_n)\cdots(1-t_3)\, t_2 + \cdots + \frac{1}{n} t_n \\ x_2(t) &= \frac{1}{2}(1-t_n)\cdots(1-t_3)\, t_2 + \frac{1}{3}(1-t_n)\cdots(1-t_3)\, x_3 + \cdots + \frac{1}{n} t_n \\ &\vdots \\ x_{n-1}(t) &= \frac{1}{n-1}(1-t_n)\, t_{n-1} + \frac{1}{n} t_n \\ x_n(t) &= \frac{1}{n} t_n \end{aligned}$$

A bit more succinctly we have

$$\begin{aligned} x_i &= \frac{1}{i}\left(\prod_{j=i+1}^{n}(1-t_j)\right) t_i + \frac{1}{i+1}\left(\prod_{j=i+2}^{n}(1-t_j)\right) t_{i+1} + \cdots \frac{1}{n-1}(1-t_n)\, t_{n-1} + \frac{1}{n} t_n \\ &= \sum_{j=i}^{n} \frac{1}{j}\left(\prod_{k=j+1}^{n}(1-t_k)\right) t_j \end{aligned}$$

where we have adopted the convention

$$\prod_{k=n+1}^{n}(1-t_k) \equiv 1$$

From this it's clear that, via multiple iterations of the change of variables formula, the integral of a polynomial $f(x)$ over $\mathcal{S}_n$ can be reduced to a sum of products of beta-integrals.

4.3. **The Reduction of $I_{n,\lambda,d}$.** Recall

$$I_{\lambda,n,d} \equiv \int_{\mathcal{S}_n} e_\lambda^{(n)}(x)\left(\prod_{1\le i<j\le n}(x_i - x_j)\right)^d dx$$

Now the factor $\prod_{1\leq i<j\leq n}(x_i-x_j)$ is just the Vandermonde determinant in n variables. Let us we write

$$\Delta^{(n)}=\prod_{1\leq i<j\leq n}(x_i-x_j)$$

so that our integral can be written a bit more succinctly as

$$I_{\lambda,n,d}=\int_{\mathcal{S}_n} s_\lambda^{(n)}\left(\Delta^{(n)}\right)^d dx$$

Let's now apply our change of variables where

$$\begin{aligned}x_i&=(1-t_n)\,t_i+\frac{1}{n}t_n\\ x_n&=\frac{1}{n}t_n\end{aligned}$$

We have

$$\begin{aligned}\left(\Delta^{(n)}\right)^d&=\left(\prod_{1\leq i<j\leq (n-1)}(x_i-x_j)\right)^d\left(\prod_{i=1}^{n-1}(x_i-x_n)\right)\\
&=\left(\prod_{1\leq i<j\leq ((n-1))}(1-t_n)\,(t_i-t_j)\right)^d\left(\prod_{i=1}^{n-1}(1-t_n)\,t_i\right)^d\\
&=(1-t_n)^{n(n-1)d/2}\left(\prod_{i=1}^{n-1}t_i\right)^d\left(\prod_{1\leq i<j\leq (n-1)}(t_i-t_j)\right)^d\\
&=(1-t_n)^{n(n-1)d/2}\left(e_{n-1}^{(n-1)}(t)\right)^d\left(\Delta^{(n-1)}(t)\right)^d\end{aligned}$$

Let

$$a\equiv\frac{t_n}{n\,(1-t_n)}$$

so that we can write

$$x_i=(1-t_n)\,(t_i-a)$$

and $t-a$ for $(t_1-a,t_2-a,\ldots,t_{n-1}-a)$.

$$\begin{aligned}e_i^{(n)}(x)&=e_{i-1}^{(n-1)}(x)\,x_n+e_i^{(n-1)}(x)\\
&=e_{i-1}^{(n-1)}((1-t_n)\,(t+a))\left(\frac{1}{n}t_n\right)+e_i^{(n-1)}((1-t_n)\,(t+a))\\
&=(1-t_n)^{i-1}\frac{1}{n}t_ne_{i-1}^{(n-1)}(t+a)+(1-t_n)^i\,e_i^{(n-1)}(t+a)\\
&=(1-t_n)^i\,ae_{i-1}^{(n-1)}(t+a)+(1-t_n)^i\,e_i^{(n-1)}(t+a)\\
&=(1-t_n)^i\left(ae_{i-1}^{(n-1)}(t+a)+e_i^{(n-1)}(t+a)\right)\end{aligned}$$

We now apply the generalized binomial expansion of Lemma 1.2 to the terms within the large parentheses

$$\begin{aligned}ae_{i-1}^{(n-1)}(t+a)+e_i^{(n-1)}(t+a)&=a\sum_{j=0}^{i-1}\binom{n-1-j}{i-1-j}a^{i-1-j}e_j^{(n-1)}(t)+\sum_{j=0}^{i}\binom{n-1-j}{i-j}a^{i-j}e_j^{(n-1)}(t)\\
&=\sum_{j=0}^{i-1}\left[\binom{n-1-j}{i-1-j}+\binom{n-1-j}{i-j}\right]a^{i-j}e_j^{(n-1)}(t)+e_j^{(n-1)}(t)\end{aligned}$$

Now

$$
\begin{aligned}
\left(\begin{array}{c} n-1-j \\ i-1-j \end{array}\right) + \left(\begin{array}{c} n-1-j \\ i-j \end{array}\right) &= \frac{(n-1-j)!}{(i-1-j)!\,(n-1-j-(i-1-j))!} + \frac{(n-1-j)!}{(i-j)!\,(n-1-j-(i-j))!} \\
&= (n-j-1)!\left(\frac{1}{(i-j-1)!\,(n-i)!} + \frac{1}{(i-j)!\,(n-i-1)!}\right) \\
&= (n-j-1)!\left(\frac{(i-j)+(n-i)}{(i-j)!\,(n-i)!}\right) \\
&= \frac{(n-j-1)!\,(n-j)}{(i-j)!\,(n-i)!}
\end{aligned}
$$

Notice that right hand side evaluates to 1 when $i = j$. [2] We can thus write

$$
ae_{i-1}^{(n-1)}(t+a) + e_i^{(n-1)}(t+a) = \sum_{j=0}^{i}\left(\frac{(n-j)\,(n-j-1)!}{(i-j)!\,(n-i)!}\right) a^{i-j} e_j^{(n-1)}(t)
$$

Thus,

$$
\begin{aligned}
e_i^{(n)}(x) &= (1-t_n)^i\left(ae_{i-1}^{(n-1)}(t+a) + e_i^{(n-1)}(t+a)\right) \\
&= (1-t_n)^i \sum_{j=0}^{i}\left(\frac{(n-j)\,(n-j-1)!}{(i-j)!\,(n-i)!}\right) a^{i-j} e_j^{(n-1)}(t) \\
&= \sum_{j=0}^{i}\left(\frac{(n-j)\,(n-j-1)!}{(i-j)!\,(n-i)!}\right)(1-t_n)^i\left(\frac{t_n}{n\,(1-t_n)}\right)^{i-j} e_j^{(n-1)}(t) \\
&= \sum_{j=0}^{i}\left(\frac{1}{n}\right)^{i-j}\left(\frac{(n-j)\,(n-j-1)!}{(i-j)!\,(n-i)!}\right)(1-t_n)^j\,(t_n)^{i-j}\, e_j^{(n-1)}(t)
\end{aligned}
$$

We now see that

$$
\begin{aligned}
e_\lambda^{(n)}(x) &= \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}^{(n)}(x) \\
&= \prod_{i=1}^{\ell(\lambda)}\left(\sum_{j_i=0}^{\lambda_i}\left(\frac{1}{n}\right)^{\lambda_i-j_i}\left(\frac{(n-j_i)\,(n-j_i-1)!}{(\lambda_i-j_i)!\,(n-\lambda_i)!}\right)(1-t_n)^{j_i}\,(t_n)^{\lambda_i-j_i}\, e_{j_i}^{(n-1)}(t)\right)
\end{aligned}
$$

We now set $k = \ell(\lambda)$ and

$$
\begin{aligned}
C_\lambda &= \left\{\mu \in \mathbb{N}^k \mid 0 \le \mu_1 \le \lambda_1\ ,\ 0 \le \mu_2 \le \lambda_2\ ,\ \ldots\ ,\ 0 \le \mu_k \le \lambda_k\right\} \\
e_\mu^{(n-1)}(t) &= \prod_{i=1}^{k} e_{\mu_i}^{(n-1)}(t) \\
c_{\lambda,\mu} &= \left(\frac{1}{n}\right)^{\lambda_i-\mu_i}\left(\frac{(n-\mu_i)\,(n-\mu_i-1)!}{(\lambda_i-\mu_i)!\,(n-\lambda_i)!}\right)
\end{aligned}
$$

Then we can write

$$
e_\lambda^{(n)}(x) = \sum_{\mu \in C_\lambda} c_{\lambda,\mu}\,(1-t_n)^{|\mu|}\,(t_n)^{|\lambda|-|\mu|}\, e_\mu^{(n-1)}(t)
$$

---

[2]with the usual convention $(0)! = \Gamma(1) = 1$.

and so

$$\begin{aligned}
I_{n,\lambda,d} &= \int_{\mathcal{S}_n} e_\lambda^{(n)}(x)\left(\Delta^{(n-1)}(x)\right)^d dx \\
&= \int_{\mathcal{S}_n}\left(\frac{1}{n}\int_0^1 (1-t_n)^{n-1}\left(e_\lambda^{(n)}(x(t))\right)\left(\Delta^{(n)}(x(t))\right)^d dt_n\right) d^{n-1}t \\
&= \frac{1}{n}\int_{\mathcal{S}_n}\left(\int_0^1 \sum_{\mu\in C_\lambda} c_{\lambda,\mu}(1-t_n)^{n-1+|\mu|+n(n-1)d/2}(t_n)^{|\lambda|-|\mu|} e_\mu^{(n-1)}(t)\left(e_{n-1}^{(n-1)}(t)\right)^d\left(\Delta^{(n-1)}(t)\right)^d dt_n\right) d^{n-1}t \\
&= \frac{1}{n}\sum_{\mu\in C_\lambda} c_{\lambda,\mu}\left(\int_0^1 (1-t_n)^{n-1+|\mu|+n(n-1)d/2}(t_n)^{|\lambda|-|\mu|} dt_n\right)\int_{\mathcal{S}_n} e_\mu^{(n-1)}(t)\left(e_{n-1}^{(n-1)}(t)\right)^d\left(\Delta^{(n-1)}(t)\right)^d d^{n-1}t \\
&= \sum_{\mu\in C_\lambda}\frac{1}{n}c_{\lambda,\mu}\beta\left(n+|\mu|+n(n-1)d/2, |\lambda|+|\mu|+1\right)\int_{\mathcal{S}_n} e_\mu^{(n-1)}(t)\left(e_{n-1}^{(n-1)}(t)\right)^d\left(\Delta^{(n-1)}(t)\right)^d d^{n-1}t \\
&= \sum_{\mu\in C_\lambda}\frac{1}{n}c_{\lambda,\mu}\beta\left(n+|\mu|+n(n-1)\nu/2, |\lambda|-|\mu|+1\right)\int_{\mathcal{S}_n} e_{\mu+de_{n-1}}^{(n-1)}(t)\left(\Delta^{(n-1)}(t)\right)^d d^{n-1}t
\end{aligned}$$

where

$$e_{\mu+de_{n-1}}^{(n-1)}(t) \equiv e_\mu^{(n-1)}(t)\left(e_{n-1}^{(n-1)}(t)\right)^d$$

We thus arrive at the following recursive formula

Proposition 4.6. *Set*

$$I_{\lambda,n,d} = \int_{\mathcal{S}_n} e_\lambda^{(n)}(x)\left(\Delta^{(n-1)}(x)\right)^d dx$$

Then

$$I_{n,\lambda,d} = \sum_{\mu\in\mathcal{C}_\lambda}\Gamma_{\lambda,\mu} I_{n-1,\mu+de_{n-1},d}$$

where

$$C_\lambda = \left\{\mu\in\mathbb{N}^k \mid 0\le\mu_1\le\lambda_1\ ,\ 0\le\mu_2\le\lambda_2\ ,\ \dots\ ,\ 0\le\mu_k\le\lambda_k\right\}$$

$$\Gamma_{\lambda,\mu} = \left(\frac{1}{n}\right)^{|\lambda|-|\mu|+1}\left(\prod_{i=1}^{\ell(\lambda)}\left(\frac{(n-\mu_i)(n-\mu_i-1)!}{(\lambda_i-\mu_i)!(n-\lambda_i)!}\right)\right)\beta\left(n+|\mu|+n(n-1)\nu/2, |\lambda|-|\mu|+1\right)$$

$$e_{\mu+de_{n-1}}^{(n-1)}(t) \equiv e_\mu^{(n-1)}(t)\left(e_{n-1}^{(n-1)}(t)\right)^d$$

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