Spherical Nilpotent Orbits and Unipotent Representations

OSU Representation Theory Seminar

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1. Introduction: The Orbit Philosophy and 3 Orbit Pictures

Normally, I like to begin a talk with a sort of where-we-are-in-the-big-picture schpeel. However, in the colloquium just a couple weeks ago, Hadi Salmasian gave a nice overview of the relevance of the orbit philosophy to the long standing and central problem of parameterizing the unitary dual. So here I'll limit myself to a couple of ancillary remarks.

Consider the group $SL(2,\mathbb{R})$. If we take the standard basis for its Lie algebra $\mathfrak{g}$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and identify, as we may, $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by use of the Killing form

$$B(Z, Z') = \frac{1}{2} trace\,(ZZ')$$

then the coadjoint orbits of $SL(2,\mathbb{R})$ fall into three basic classes; according to whether the Casimir function

$$B(Z, Z) = h^2 + xy$$

is positive, negative or zero.(Here $Z \equiv x * X + h * H + y * Y$.) The nature of these orbits becomes a little clear if we adopt a basis for which B is diagonal, setting

$$\begin{aligned} Z_0 &= X - Y \\ Z_2 &= X + Y \\ Z_3 &= H \end{aligned}$$

we find

$$B(Z, Z) = -z_0^2 + z_2^2 + z_3^2$$

That is, the invariant bilinear form on $\mathfrak{g}$ looks like the Lorentz metric on $\mathbb{R}^{2,1}$. And, in fact, the orbit structure of $\mathfrak{g}$ looks like that of (2+1)-dimensional Minkowki spacetime; thinking of z_0 as the "temperal coordinate" and z_1 and z_2 as the "spatial coordinates". The three orbit classes are

- $B(Z, Z) > 0$. The hyperbolic orbits.
- $B(Z, Z) < 0$. The elliptic orbits
- $B(Z, Z) = 0$ The nilpotent orbits.

For the first two orbit types there are more or less uniform methods of attaching unitary representations; respectively, (unitary) parabolic induction and cohomological parabolic induction The point of the current talk is to outline a uniform method for attaching unitary representations to a family of spherical nilpotent orbits.

Now in fact there are many ways of attaching a representation to an orbit. But they fall into two basic classes, and within each class there are several more-or-less canonical instances.

- **Quantization Methods:** by which I mean, the direct construction of π from geometric data attached to an orbit.
  - parabolic induction for hyperbolic orbits
  - cohomological parabolic induction for elliptic orbits
  - ??? for nilpotent orbits (no uniform construction is known)
- **Dequantization Methods:** by which I mean the identification of a particular orbit via some limit, contraction, or grading process.
  - The characteristic variety of the $Ann(\pi) \subset U(\mathfrak{g})$, a $G_{\mathbb{C}}$-orbit in $\mathcal{N}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}^*$
  - The wave front set of π, a $G_{\mathbb{R}}$ orbit in $\mathfrak{g}_R = \mathcal{N}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}^*$
  - The associated variety of π, a $K_{\mathbb{C}}$-orbit in $\mathcal{N}_{\mathfrak{p}} \subset (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}) \approx \mathfrak{p}_{\mathbb{C}}$

Now actually for the problem of parameterizing the unitary dual, only the quantization methods are directly relevant. In fact, the dequantization methods produce only nilpotent orbits, which of course is only a finite set. (Thus, as far as the parameterization of the unitary dual goes dequantization methods provide only a crude partitioning of the unitary dual into a finite number of subsets.[1]

However, since it is the nilpotent orbits that stand as the last to succumb to a quantization scheme, and because in the cases where one can construct representations from a nilpotent orbits, dequantization methods take you invariably back to the same nilpotent orbit, it not so unnatural to think of a quantization method for nilpotent orbits as begining in any one of the three (tightly related) nilpotent orbits. In this talk, I shall propose a uniform method for attaching unitary representations to certain families of spherical nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$. A little more explicitly, I shall propose an explicit program for analyzing the signature characters and reducibility of certain degenerate principal series representations corresponding to the families of spherical nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ that arise from sequences of strongly orthogonal noncompact weights.

2. Sahi's Construction

I'll begin by reviewing S. Sahi's construction[2] of certain families of unipotent representations occurring the simple Lie groups associated to simple real Jordan algebras; as this provides the basic template for the methodology I am proposing.

2.1. **Apparatus.**

Sahi begins with a simple Lie group G with Lie algebra $\mathfrak{g}$, maximal compact subgroup K, and Cartan involution θ for which G has a parabolic subgroup $P = MAN$ with the following two properties

(i) the nilradical N is abelian;
(ii) P is conjugate to θN .

For such groups the Lie algebra $\mathfrak{n}$ of G has the structure of simple real Jordan algebra and, in fact, conversely, given a simple real Jordan algebra $\mathfrak{n}$, a Lie group G with the properties (i) and (ii) above arises naturally as the conformal group of $\mathfrak{n}$ via a standard construction due to Koecher and Tits.

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[1]Moreover, refinements such as characteristic cycles, do little to improve the lack of injectivity inherent to dequantization methods

[2]S. Sahi, *Jordan algebras and degenerate principal series*, J. reine angew. Math **462** (1995), 1-18.

In this setting Sahi considers a continuous family of spherical principal series representations

$$I(s) = Ind_{MAN}^{G}\left(1 \otimes e^{s\nu} \otimes 1\right) \quad , \quad s \in \mathbb{R}$$

where e^{ν} is the character of A corresponding to, on the one hand, the half sum of the positive restricted roots w.r.t. $\mathfrak{a} = Lie(A)$, or on the other hand, the determinant of action of $L = MA$ on $\mathfrak{n} = Lie(N)$. The unipotent representations of interest occur as certain unitarizable constituents that "pop out" of the generically irreducible representations $I(s)$ are certain discrete values of the parameter s (i.e., at certain "reduction points").

We point out that the hypotheses (i) and (ii) above, besides landing us squarely in the Jordan algebra setting have two crucial representation theoretical consequences for the spherical principal representations $I(s)$. For (i) guarantees

(i′) the K-types of $I(s)$ have multiplicity one.

a simplification that's essential for Sahi's analysis; and (ii) guarantees that

(ii′) each irreducible constituent of $I(s)$ carries an invariant Hermitian form,

which, of course, is a prerequisite for unitarity.

However, before describing Sahi's methodology I should point out three other salient features of Sahi's setup.

(iii) If $\{e_1, \ldots, e_n\}$ is a complete set of primitive idempotent in $\mathfrak{n}$, thought of a Jordan algebra, then the $\{e_i\}$ thought of as elements of $\mathfrak{g}$ are strongly orthogonal. Moreover, to each primitive idempotent e_i there is a corresponding TDS (three-dimensional $\mathfrak{sl}_2$ subalgebra) $\mathfrak{s}_i$, and $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ if $i \neq j$.
(iv) For each non-negative integer k, $I(r + 2k)$ has an irreducible finite-dimensional spherical subrepresentation. (Here r is just the ρ-shift associated to normalized parabolic induction.)
(v) The norm function on the associated Jordan algebra gives rise to certain invariant differential operators D_m that intertwines $I(m)$ with its dual $I(-m)$. (These are the Kostant-Sahi Capelli operators.)

2.2. **Methodology.**

2.2.1. *The Hermitian Form on K-types.* Let $\mathfrak{t}_0$ be a maximal compact subalgebra of $\mathfrak{l} \cap \mathfrak{k}$ and let $\mathfrak{t}_1$ be a maximal compact subalgebra of the orthogonal complement of $\mathfrak{l} \cap \mathfrak{k}$ in $\mathfrak{k}$, so that $\mathfrak{t} = \mathfrak{t}_0 + \mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{k}$. Let $n = \dim \mathfrak{t}_1$. Let $\Sigma = \Sigma(\mathfrak{t}_1, \mathfrak{k})$ be the restricted root system of $\mathfrak{k}$ with respect to $\mathfrak{t}_1$. For ease of exposition, we describe how Sahi's analysis works out in the case when Σ is of Cartan type C_n. (The other possibilities are $\Sigma = A_{n-1}$ and D_n, and for these cases the methodology is the same, up to slight discrepancies in the formulas.) Sahi fixes a positive system for Σ and adopts a basis $\{\gamma_1, \ldots, \gamma_n\}$ for $\mathfrak{t}_1^*$ so that the simple roots of Σ are

$$\frac{1}{2}(\gamma_1 - \gamma_2)\,,\ \frac{1}{2}(\gamma_2 - \gamma_3)\,,\ \ldots\ ,\ \frac{1}{2}(\gamma_{n-1} - \gamma_n)\,,\ \gamma_n$$

and it turns out that each γ_i is the restriction to $\mathfrak{t}_1$ of an extremal weight of the representation of K on $\mathfrak{p}$, and that (the highest weights of) the K-types of $I(s)$ are

$$\mathcal{S} = \{\mu = a_1\gamma_1 + \cdots + a_n\gamma_n \in span_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_n) \mid a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\} \quad .$$

Let us write V for the $(\mathfrak{g}, K)$-module of $I(s)$ (i.e. the subspace of K-finite vectors), and denote by V_μ the (unique) irreducible summand of V corresponding to the K-type $\mu \in \mathcal{S}$. We denote by $\langle\cdot,\cdot\rangle_s$ the $\mathfrak{g}$-invariant Hermitian form on V inherited from $I(s)$ (unique up to scalar, but dependent upon s), and by $\langle\cdot,\cdot\rangle_K$ the (unique up to scalar) positive-definite Hermitian form arising from the realization of V as $L^2(K/(L \cap K))_{K-finite}$ Since V is multiplicity free, the restriction of $\langle\cdot,\cdot\rangle_s$ to any particular K-type V_α

must coincide with the restriction of $\langle\cdot,\cdot\rangle_K$ to V_α up to an overall scalar factor. And since $V_\alpha \perp V_\beta$ if $\alpha \neq \beta$, we can write

$$\langle\cdot,\cdot\rangle_s = \bigoplus_{\alpha\in\mathcal{S}} \varepsilon_\alpha(s)\,\langle\cdot,\cdot\rangle_K|_{V_\alpha}$$

In this light, **a unitarizable submodule of $I(s)$ will manifest itself as a g-invariant set of K-types for which the scalar factors $\varepsilon_\alpha(s)$ are either all positive or all negative**.

2.2.2. *The local transition functions $c_{\alpha,i}(s)$ and $d_{\alpha,i}(s)$.* To get a handle on the relative signs K-type weight functions $\varepsilon_\alpha(s)$ and indeed to identify $\mathfrak{g}$-invariant subsets of K-types, Sahi examines certain "transition functions" between neighboring K-types.[3] Suppose a K-types α is connected to a different K-type β by an element $X \in \mathfrak{p}$. A little more explicitly, suppose there is a $v_\alpha \in V_\alpha$, $v_\beta \in V_\beta$ and elements $X, \overline{X} \in \mathfrak{p}$ such that

$$v_\beta = \pi_s(X)\,v_\alpha \quad , \quad v_\alpha = \pi_s\left(\overline{X}\right)v_\beta \quad .$$

Then the $\mathfrak{g}$-invariance of $\langle\cdot,\cdot\rangle_s$ requires

$$\langle\pi_x(X)\,v_\alpha, v_\beta\rangle_s = -\left\langle v_\alpha, \pi_s\left(\overline{X}\right)v_\beta\right\rangle_s$$

or

$$(*) \qquad \varepsilon_\beta(s)\,\langle\pi_s(X)\,v_\alpha, v_\beta\rangle_K = -\varepsilon_\alpha(s)\left\langle v_\alpha, \pi\left(\overline{X}\right)v_\beta\right\rangle_K$$

Now note that we can, without loss of generality, take both v_α and v_β are normalized so that

$$\langle v_\alpha, v_\alpha\rangle_K = 1 = \langle v_\beta, v_\beta\rangle_K$$

and suppose further that the $\langle\cdot,\cdot\rangle_K$-orthogonal projections of $\pi_s(X)\,v_\alpha$ onto v_β is $c_{\dot\alpha,\beta,X}(s)\,v_\beta$, and similarly the orthogonal projection of $\pi_s\left(\overline{X}\right)v_\beta$ onto v_α is $d_{\alpha,\beta,X}(s)\,v_\alpha$. Then condition (*) becomes

$$\varepsilon_\beta(s)\,c_{\alpha,\beta,X}(s) = -\varepsilon_\alpha(s)\,d_{\alpha,\beta,X}(s)$$

From this it is clear that the occurrence of "signature flips" in the $\langle\cdot,\cdot\rangle_s$ would be revealed by getting a handle on all the transition functions between neighboring K-types.

Well, unfortunately, finding all the functions $c_{a,\beta,X}(s)$ and $d_{\alpha,\beta,X}(x)$ is way, way too much to hope for. N.B., that even after fixing the K-types α and β, there will be many possible choices of $v_\alpha \in V_\alpha$, $v_\beta \in V_\beta$ and $X, \overline{X} \in \mathfrak{p}$. Moveover, without some additional organizing principles, the transition functions cannot be used to identify submodules for which the sign of the local signatures are constant. For even if you knew *all* the transition functions between V_α and V_β are zero, you would not be cannot conclude that V_α and V_β can not be connected by an element of $U(\mathfrak{g})$ via an alternate (albeit non-direct); e.g. as in

$$\begin{array}{ccccc} V_\alpha & & \not\rightleftarrows & & V_\beta \\ & \searrow & & \nearrow & \\ & & V_\gamma & & \end{array}$$

We shall see below that Sahi's setup permits the exploitation of a small, tractable family of transitions functions to detect both reduciblility and unitarity.

Observation 2.1. *Let V_α be a K-type in V. Then the K-types V_β that are directly reachable from V_α via $\mathfrak{g}$ have $\beta = \alpha \pm \gamma_i$ for some $i \in \{1,\ldots,n\}$.*

The K-types of V that are directly reachable from V_α must also be K-types in

$$\mathfrak{p}\otimes V_\alpha \quad .$$

[3] Here I use the term "transition functions" by analogy with the notion of transition matrix in quantum mechanics; not by analogy with differential geometry.

As is well known,[4] the possible highest weights in such a tensor product must be of the form $\alpha+\mu$ with μ a weight of $\mathfrak{p}$. The weights (really, the restricted weights of $\mathfrak{p}$, but no matter ...) of $\mathfrak{p}$

$$\{\pm\gamma_i \mid i=1,\dots,n\} \cup \left\{\frac{1}{2}\left(\pm\gamma_i \pm \gamma_j\right) \mid 1\le i<j\le n\right\}$$

However, it is clear that since the $\alpha \in \mathcal{S}$ must be integral linear combinations of the γ_i, only highest weights of the form $\alpha \pm \gamma_i$ can be in both V and $\mathfrak{p}\otimes V_\alpha$. $\square$

Theorem 2.2 (Kumar)**.** *Let $\mathfrak{g}$ finite-dimensional semisimple Lie algebra, endowed with a chosen Cartan algebra $\mathfrak{h}$, positive system $\Delta^+(\mathfrak{h},\mathfrak{g})$ and Weyl group $W(\mathfrak{h},\mathfrak{g})$. Let λ,μ be any pair of dominant weights for $\Delta^+(\mathfrak{h},\mathfrak{g})$ and consider the tensor product $V_\lambda\otimes V_\mu$ of the corresponding finite-dimensional represenations. Then*

- *Whenever $w\in W(\mathfrak{h},\mathfrak{g})$ is such that $\lambda+w\mu$ is dominant, $V_\lambda\otimes V_\mu$ has an irreducible subrepresentation $(V_\lambda\otimes V_\mu)_{\lambda+w\mu}$ of highest weight $\lambda+w\mu$. Moreover, this subrepresentation occurs in $V_\lambda\otimes V_\mu$ with multiplicity exactly one..*
- *If v_λ is a highest weight vector for V_λ and $v_{w\mu}$ is a weight vector in the $w\mu$ weight space of V_μ, then the vector $v_\lambda\otimes v_{w\mu}$ has a non-zero projection onto the highest weight space of $(V_\lambda\otimes V_\mu)_{\lambda+w\mu}$.*

From the fact that the $\{\gamma_1,\dots,\gamma_n\}$ are all extremal weights of the representation of K on $\mathfrak{p}$, we can now infer

Observation 2.3. *For any $\alpha\in\mathcal{S}$, let $P_\alpha : V\to V_\alpha$ denote the corresponding K-equivariant orthogonal projection of V onto the K-type with highest weight α. Let v_α, v_β be the (unique up to scalars) highest weight vectors of, respectively, two different K-typesV_α, V_β in V. Then V_α is directly connected to V_β; i.e., there exists an element $X\in\mathfrak{g}$ such that $ad(X)\,V_\alpha\cap V_\beta\neq\{0\}$, if and only if there is an extremal weight γ_i of $\mathfrak{p}$ such that $\beta=\alpha+\gamma_i$ and*

$$P_\beta\pi_s(X_i)\,v_\alpha = c_{\alpha,i}(s)\,v_\beta \neq 0$$

where $0\neq X_i\in\mathfrak{p}_{\gamma_i}$.

Suppose now for each K-type V_λ, $\lambda\in\mathcal{S}$, we choose a highest weight vector v_λ normalized so that

$$\langle v_\lambda, v_\lambda\rangle_K = 1 \quad .$$

Then, by the preceding observations, if a K-type V_α is (directly) connected to a K-type V_β, we must have $\beta=\alpha\pm\gamma_i$ for some $i\in\{1,\dots,n\}$. In fact, we can assume (by if necessary relabeling α and β) that $\beta=\alpha+\gamma_i$, and that

$$P_\beta\left(\pi_s(X_i)\,v_\alpha\right) = c_{\alpha,i}(s)\,v_\beta$$
$$P_\alpha\left(\pi_s\left(\overline{X}_i\right)v_\beta\right) = d_{\alpha,i}(s)\,v_\alpha$$

In fact, we can refine the choice of $X_i\in\mathfrak{p}_{\gamma_i}$ and the normalizations of v_α and v_β so that $c_{\alpha,i}(s)$, $d_{\alpha,i}(s)$ are real and the criteria (*) becomes

$$\frac{c_{\alpha,i}(s)}{d_{\alpha,i}(s)} = \frac{\varepsilon_\beta(s)}{\varepsilon_\alpha(s)}$$

In summary, as far as the signature character of V goes, it suffices to understand the transition functions between highest weight vectors of neighboring K-types.

2.2.3. ***Consequences of the realization*** $I(s)=Ind_{MAN}^{G}\left(1\otimes e^{\nu s}\otimes 1\right)$. It follows from the fact that V is the underlying space of the realization of a spherical parabolically induced representation in the *compact picture*, that $\pi_s(X)$ is an affine function of s for any $X\in\mathfrak{g}$. This implies, in particular, that the transition

[4] See, for example, Execise 13, on page 111 in Knapp's book *Representation Theory of Semisimple Lie Groups.*

functions $c_{\alpha,i}(s)$ and $d_{\alpha,i}(s)$, are *affine* functions of the inducing parameter s. Thus, for example, we can write

$$c_{\alpha,i}(s) = A_{\alpha,i}(B_{\alpha,i} - s/2) \tag{1}$$

Another consequence of the fact that V is the underlying space of the compact picture realization of a spherical induced representation is the following identity

$$\pi_{t+t'}(X)(vu) = (\pi_t(X)u)v + u(\pi_{t+r}(X))v$$

This identity leads to the following lemma.

Lemma 2.4. *For $i = 1, \ldots, n$, let $\mu_i = \gamma_1 + \cdots + \gamma_i$. Suppose α and $\alpha + \gamma_i$ are in $\mathcal{S}$. If $i \leq j$, then*

$$c_{\alpha+\mu_j,i}(t+2) = k_i c_{\alpha,i}(t) \tag{2}$$

where k_i is a non-zero constant (independent of t).

2.2.4. ***Consequences of the Capelli Identity.*** In an earlier paper Kostant and Sahi found a *Capelli* operator D intertwining the representations $I(2)$ and $I(-2)$, and established that on a K-type V_α,, D acts by the scalar

$$D|_{V_\alpha} = \prod_{j=1}^{n}\left(a_j + r_j - \frac{1}{2}\right)\left(a_j + r_j + \frac{1}{2}\right)\cdot \mathbf{1}_{V_\alpha} \tag{**}$$

if $\alpha = a_1\gamma_1 + \cdots + a_n\gamma_n$. Applying the intertwining property of D to the current setting

$$\pi_{-2}(X_i)Dv_\alpha = D\pi_2(X_i)v_\alpha$$

and applying (**), Sahi obtains

$$\left(a_i + r_i - \frac{1}{2}\right)\left(a_i + r_i + \frac{1}{2}\right)c_{\alpha,i}(-2) = \left(a_i + r_i + \frac{1}{2}\right)\left(a_i + r_i + \frac{3}{2}\right)c_{\alpha,i}(2)$$

Assuming $c_{\alpha,i}(s)$ is not identically zero, (1) and (3) imply

$$B_{\alpha,i} = a_i + r_i + \frac{1}{2} \tag{3}$$

From (1), (3), one can now, nearly, deduce the following lemma and corollary.

Lemma 2.5. *Suppose α and $\alpha + \gamma_i$ are n $\mathcal{S}$, with $\alpha = \sum_{i=1}^{n} a_i\gamma_i$. If the transition function $c_{\alpha,i}(s)$ is not identically zero, then $c_{\alpha,i}(s)$ is a non-zero multiple of $\left(a_i + r_i + \frac{1}{2}(1-s)\right)$.*

Corollary 2.6. *Suppose α and $\alpha + \gamma_i$ are n $\mathcal{S}$, with $\alpha = \sum_{i=1}^{n} a_i\gamma_i$ and the transition function $c_{\alpha,i}(s)$ is not identically zero. If we normalize v_α and $v_{\alpha+\gamma_i}$ so that*

$$c_{\alpha,i}(s) = (a_i + r_i + (1-s)/2)$$

Then

$$d_{\alpha,i}(s) = (a_i + r_i + (1+s)/2)\frac{\langle v_{\alpha+\gamma_i}, v_{\alpha+\gamma_i}\rangle_K}{\langle v_\alpha, v_\alpha\rangle_K}$$

2.2.5. *Consequences of (iii) and (iv) and the formulas for $c_{\alpha,i}(s)$ and $d_{\alpha,i}(s)$.* The explicit determination of the transition functions is thus reduced to verifying that the transitions functions $c_{\alpha,i}(s)$ can not be identically zero. This fact Sahi proves as follows

- The fact that $I(2)$ has a irreducible spherical subrepresentation F_1 with K-types 0 and

  $$\mu_i = \gamma_1 + \cdots \gamma_i \qquad , \qquad 1 \leq i \leq n$$

  implies that the transitions functions

  $$\text{for } 1 \leq i \leq n \quad , \quad c_{\mu_{i-1},i}(s) \not\equiv 0 \quad \text{as a function of } s\ .$$

  Otherwise, F_1 itself would be reducible.

- Lemma 2.4 above, the strong orthogonality of the X_i and some diagram chasing then permits Sahi to infer from the non-triviality of the $c_{\mu_{i-1},i}(s)$ that

$$\text{If } \alpha \text{ and } \alpha+\gamma_i \text{ are in } \mathcal{S}, \text{ then } c_{\alpha,,i}(s) \not\equiv 0 \quad \text{as a function of } s$$

and so we can replace Corollary 4.3 with

Theorem 2.7. *Suppose* $\alpha = \sum_{i=1}^{n} a_i\gamma_i$ *and both* α *and* $\alpha+\gamma_i$ *are n* $\mathcal{S}$. *Then, we can normalize* v_α *and* $v_{\alpha+\gamma_i}$ *so that*

$$c_{\alpha,i}(s) = (a_i + r_i + (1-s)/2)$$

and

$$d_{\alpha,i}(s) = (a_i + r_i + (1+s)/2)\,\frac{\langle v_{\alpha+\gamma_i}, v_{\alpha+\gamma_i}\rangle_K}{\langle v_\alpha, v_\alpha\rangle_K} \quad .$$

3. Generalization to Arbitrary Semisimple Groups

Ostensibly, Sahi's construction is highly dependent on the Jordan algebra setting from which it arose. In particular, circumstances (i) and (ii) effectively restrict G to the class of groups associated with real semisimple Jordan algebras. Circumstance (iv) also seems very much a relic of the Jordan algebra setting. The thesis and whole point of this talk is that the real underpinnings of Sahi's construction; i.e. circumstances (i'), (ii'), (iii), (iv) and even (v) can be replicated for any connected semisimple group G via the notion of sequences of non-compact roots I introduced in a talk last spring.

Let me recall that business. For simplicity, let me assume that G is not of hermitian symmetric type. Fix a maximal compact subgroup K, Cartan involution θ, and Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ of the complexified Lie algebra of G. The representation of K on $\mathfrak{p}$ is irreducible. Let γ_1 be the highest weight of $\mathfrak{p}$ with respect to some Cartan algebra $\mathfrak{t}$ of $\mathfrak{k}$ and some choice of positive system $\Delta^+(\mathfrak{t},\mathfrak{k})$. Starting with γ_1 we construct a sequence $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ of strongly orthogonal noncompact weights according to the following inductive prescription:

- γ_i lies in the Weyl orbit of γ_1
- $\gamma_1 + \cdots + \gamma_i$ is a dominant weight with respect to $\Delta^+(\mathfrak{t},\mathfrak{k})$
- $X_i \in \mathfrak{p}_{\gamma_i}$ is strongly orthogonal to every $X_j \in \mathfrak{p}_j$, $j < i$. (This means that neither $\gamma_i+\gamma_j$ nor $\gamma_i-\gamma_j$ is a $\mathfrak{t}$-weight of $\mathfrak{g}$.)

Now let $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ be such a (maximal) sequence. Then in each (1-dimensional) weight space $\mathfrak{p}_{\gamma_i}$ we can choose a representative nilpotent noncompact element x_i, and construct an associated normal triple $\{x_i,h_i,y_i\}$; that is to say additional elements $h_i \in \mathfrak{t}$, $y_i \in \mathfrak{p}_{-\gamma_i}$ such that

$$[h_i,x_i] = 2x_i \quad , \quad [h_i,y_i] = -2y_i \quad , \quad [x_i,y_i] = h_i$$

In fact, we can choose $\{x_i,h_i,y_i\}$ so that

$$y_i = \overline{x_i}$$
$$h_i \in i\mathfrak{t}_{\mathbb{R}}$$

and we do so because this becomes useful later on. Morever, because the x_i are all strongly orthogonal the corresponding $\mathfrak{sl}_2$ subalgebras of $\mathfrak{g}$

$$\mathfrak{s}_i = span_{\mathbb{C}}(x_i,h_i,y_i)$$

will all commute with each other. Let

$$\mathfrak{t}_1 = span_{\mathbb{C}}(h_1,\dots,h_n)$$

and let $\mathfrak{t}_0$ be the orthogonal complement of $\mathfrak{t}_1$ in $\mathfrak{t}$.

Theorem 3.1 (B). *Let*

$$Y_i = y_1 + \cdots + y_i$$

Then the representation of $K_{\mathbb{C}}$ carried by the regular functions on the closure of $K_{\mathbb{C}} \cdot Y_i$ is multiplicity free and its K-types μ are exactly

$$\mathcal{S}_i = \{\mu = a_1\gamma_1 + \cdots + a_i\gamma_i \in span_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_i) \mid a_1 \geq a_2 \geq \cdots \geq a_i \geq 0\}$$

if $i < n$. If $i = n$ then one has

$$\mathcal{S}_n = \{\mu = a_1\gamma_1 + \cdots + a_n\gamma_n \in span_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_i) \mid a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n\}$$

if the restricted root system $\Sigma(\mathfrak{t}_1, \mathfrak{k})$ is of type A_{n-1} or

$$\mathcal{S}_n = \{\mu = a_1\gamma_1 + \cdots + a_n\gamma_n \in span_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_i) \mid a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0\}$$

if the restricted root system is of type D_n, or

$$\mathcal{S}_n = \{\mu = a_1\gamma_1 + \cdots + a_n\gamma_n \in span_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_i) \mid a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n \geq 0\}$$

if $\Sigma(\mathfrak{t}_1, \mathfrak{k}) \neq A_{n-1}, D_n$.

3.1. **Cayley transform.** From here on we restrict our attention to the "quasi-principal" TDS $\{X, H, Y\} = \{X_n, H_n, Y_n\}$ and the big orbit $K_{\mathbb{C}} \cdot Y$. Set

$$\begin{aligned}
\widetilde{X} &= \frac{1}{2}(X + Y - iH) \\
\widetilde{H} &= -i(X - Y) \\
\widetilde{Y} &= \frac{1}{2}(X + Y + iH)
\end{aligned}$$

then $\left\{\widetilde{X}, \widetilde{H}, \widetilde{Y}\right\}$ is a Cayley triple in $\mathfrak{g}_{\mathbb{R}}$, that is to say, a standard triple in $\mathfrak{g}_{\mathbb{R}}$ such that

$$\begin{aligned}
\theta\widetilde{X} &= -\widetilde{Y} \\
\theta\widetilde{Y} &= -X \\
\theta\widetilde{H} &= -\widetilde{H}
\end{aligned}$$

3.2. **Spherical induced representation.** Since $\widetilde{H}$ is a semisimple element of $\mathfrak{p}_{\mathbb{R}}$, we can use it to construct a certain parabolic subgroup $P = MAN$ of G as well as a certain character of A. This construction goes as follows.

We now define

$$\begin{aligned}
\mathfrak{n} &= \text{direct sum of positive eigenspaces of } ad\left(\widetilde{H}\right) \text{ in } \mathfrak{g}_{\mathbb{R}} \quad , \\
\mathfrak{l} &= 0\text{-eigenspace of } ad\left(\widetilde{H}\right) \text{ in } \mathfrak{g}_{\mathbb{R}} \quad , \\
\mathfrak{a} &= Z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbb{R}} \quad , \\
\mathfrak{m} &= \text{orthogonal complement of } \mathfrak{a} \text{ in } \mathfrak{l}
\end{aligned}$$

and then set

$$\begin{aligned}
M &= Z_K(\mathfrak{a}) \exp(\mathfrak{m}) \quad , \\
A &= \exp(\mathfrak{a}) \quad , \\
N &= \exp(\mathfrak{n}) \quad .
\end{aligned}$$

Then $P = MAN$ is a (Langlands decomposition of a) parabolic subgroup of G. Moreover, it happens that

$$span_{\mathbb{R}}\left(\widetilde{h}_1, \ldots, \widetilde{h}_n\right) \subseteq \mathfrak{a} \quad .$$

Now let ν be the element of the real dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ such that

$$\nu(h) = B_0\left(\widetilde{H}, h\right) \qquad \forall\ h \in \mathfrak{a}_0 \quad ,$$

where $B_0(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}_{\mathbb{R}}$ restricted to $\mathfrak{a}$.

Lemma 3.2. *The K-types of*

$$I(s) = Ind_{MAN}^{G}\left(1 \otimes e^{\nu s} \otimes 1\right)$$

coincide with those of $\mathbb{C}\left[\overline{K_{\mathbb{C}} \cdot Y}\right]$. In particular, if V is the realization of $I(s)$ in the compact picture then

$$V = \bigoplus_{\mu \in \mathcal{S}} V_\mu$$

with each K-type occuring with multiplicity one and in the $\mathbb{Z}$-span of the weights $\gamma_i \in \Gamma$.

Proof. This is essentially a verification that the algebraic Frobenious reciprocity argument used to determine the K-types in $\mathbb{C}\left[\overline{K_{\mathbb{C}} \cdot Y}\right]$ is compatible with the analytic Frobenious reciprocity argument used to identify the K-types of $I(s)$. The key to this is the observation that $\mathfrak{m}$ is preserved by the Cayley transform.

■ This lemma provides us with a replication of circumstance (i').

To replicate circumstance (ii'), we consider

$$w = \exp\left(\frac{\pi}{2}\left(\widetilde{X} - \widetilde{Y}\right)\right) \in K \quad .$$

It then happens that

Lemma 3.3. *With $P = MAN$, $\nu \in \mathfrak{a}^*$ and $w \in N_K(\mathfrak{a})$ defined as above, we have*

- $w \in N_K(\mathfrak{a})$;
- $wPw^{-1} = \overline{P}$*, the parabolic opposite to* P
- $Ad^*(w)\,\nu = -\nu$

I now quote a fundamental result of Knapp and Zuckerman

Theorem 3.4. *Suppose*

$$w \in N_K(\mathfrak{a}) \quad , \quad wPw^{-1} = \overline{P} \quad , \quad Ad^*(w)\,\nu = -\nu$$

then the Langlands quotient of $Ind_{MAN}^{G}\left(1 \otimes e^{\nu} \otimes 1\right)$ carries a non-degenerate hermitian form.

Since for generic s spherically induced representations are irreducible, Lemma , Theorem and a Jantzen filtration argument gives us a invariant non-degenertate hermitian form on each irreducible submodule of

$$I(s) = Ind_{MAN}^{G}\left(1 \otimes e^{s\nu} \otimes 1\right) \quad .$$

■ We thus arrive at a replication of circumstance (ii').

■ As for circumstance (iii), well, the existence of the strongly orthogonal non-compact root vectors is the foundation of our whole setup.

We're now up to circumstance (iv), the appearance of finite-dimensional subrepresentations within $I(s)$ for certain s.

Lemma 3.5. *Let $r \in \frac{1}{2}\mathbb{N}$ be defined by*

$$\frac{1}{2}\sum_{\alpha \in \Sigma(\mathfrak{t}_1, \mathfrak{g})} m_\alpha \alpha = r\left(\gamma_1 + \cdots + \gamma_n\right)$$

Then, for any integer k, $I(2k + r)$ contains a irreducible finite-dimensional spherical representation of $\mathfrak{g}$ of highest weight $k(\gamma_1 + \cdots + \gamma_n)$.

Proof. In Helgason, *Groups and Geometric Analysis*, one can find the following statement.[5]

Theorem 3.6. *Let π be an irreducible finite-dimensional representation of G. Then the following statements are equivalent.*

(i) *π has a non-zero K-fixed vector.*
(ii) *The highest weight ν of π vanishes on $\mathfrak{t}_0 \subset \mathfrak{m}$, and the restriction of ν to $\mathfrak{a}$ is such that*

$$\frac{\langle \nu, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \textit{ for every restricted root } \beta \in \mathbb{Z}$$

It is easy to see that the weights $k\mu_n = k(\gamma_1 + \cdots + \gamma_n)$ satisfy the conditions (ii), and so there exists a finite-dimensional spherical representation of highest weight $k\mu_n$. Moreover, if ϕ_ν is the highest weight vector for such a representation π_k and ϕ_K is the corresponding spherical vector, it is easy to see that the matrix element functions

$$\phi(g) = \langle \pi(g) \phi_\nu, \phi_K \rangle$$

transform as the irreducible finite-dimensional representation contragredient to π, and moreover

$$\phi(g) \in I(2k + r) \quad .$$

These last two statements follow from easy calculations that are carried out very explicitly in Knapp's book (Representation Theory of Semisimple Groups, §9.6)

■ We thus have a replication of circumstance (iv).

Lastly, we turn to task of replicating circumstance (v) in our generalized setting. Again our goal is not so much to replicate the Jordan algebraic Capelli operator *per se*, but rather to find a representation theoretical construct that provides the same functionality. And here again, once we figure out what we're looking for, we find the problem has been solved several times over in the literature.

So what are we looking for? Well, ostensibly, the Kostant-Sahi Capelli operators are born from the Jordan norm on $\mathfrak{n}$. This form provides a homogeneous polynomial on $\mathfrak{n}$ which by duality leads to a certain constant coefficient operator on $C^\infty(\mathfrak{n})$. Interpreting the latter as the representation space for $I(s)$, this operator becomes a certain L-quasi-invariant differential operator intertwining $I(1)$ with $I(-1)$. Cayley transforming to the compact picture, yields a K-invariant operator on K/M that continues to intertwine $I(1)$ and $I(-1)$. Then via the Harish-Chandra homomorphism Kostant and Sahi obtain the formula for the eigenvalue of D on a K-type V_α. That seems a lot to ask for.

Not so. First of all, from an old paper of Kostant (*Verma modules, and the existence of quasi-invariant differential operators*) there is a natural duality (corresponding to differentiation functions in $Ind_{MAN}^G(1 \otimes e^\nu \otimes 1)$ at the identity), that reduces the problem of finding differential operators intertwining two spherical principal series representations to the problem of finding intertwining maps between generalized Verma modules of scalar type. The latter gadgets are defined as follows.

Definition 3.7. *Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ and E a 1-dimensional representation of $\mathfrak{p}$. The corresponding* generalized Verma module of scalar type *is*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E \quad .$$

Suppose now that E and E' are two 1-dimensional $\mathfrak{q}$-modules, having non-zero elements e and e' of weight λ and λ' with respect to the $\mathfrak{a}$-part of $\mathfrak{q}$. ($\mathfrak{q} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is the usual Langlands decomposition). Then every $\mathfrak{g}$-homomorphism

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E' \quad \to \quad U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$$

[5] pg. 535.

is determined by the image $\phi(1 \otimes e')$. Via the Poincare-Birkhoff-Witt theorem we can write

$$\phi(1 \otimes e') = u \otimes e$$

for some $u \in U(\overline{\mathfrak{n}})$. Since E and E' are 1-dimensional and $\mathfrak{m}$ is semisimple, it follows that $u \in U(\overline{\mathfrak{n}})^{\mathfrak{m}}$. Morever, since ϕ must be in particular an $\mathfrak{a}$-homomorphism, u must have weight $\lambda' - \lambda$. Hence, u must be an $\mathfrak{l}$-semi-invariant in $U(\overline{\mathfrak{n}})$. Moreover, u has to be annihilated by $\mathfrak{n}$ since $1 \otimes e'$ is annihilated by $\mathfrak{n}$. Finally, just as in the case of ordinary Verma modules (in fact, it follows from this case since every generalized Verma module is a quotient of an ordinary Verma module), $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E'$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ have to have the same infinitesimal character, and this leads to the stipulation that there exist a Weyl group element such that

$$w(\lambda + \rho) = \lambda' + \rho \tag{***}$$

How are we going to find such a u? Easy. Remember our finite-dimensional subrepresentation $F_\mu \subset I(1)$. It is easy to see that in the noncompact picture this correspond to a certain $\mathfrak{g}$-invariant set of polynomials on $\overline{\mathfrak{n}}$. Moreover, F_μ is spherical, and its highest weight vector ψ is $\mathfrak{m}$-invariant and $\mathfrak{l}$-semi-invariant.It follows that the image of ψ in $U(\mathfrak{n})$ via the symmeterizer map will have all the properties we need for u except possibly (***). But it's just as easy to check that (***) holds as well - using the same Weyl group element w we used to establish the existence of the hermitian form on $I(s)$.

There is a even easier way to get one's hands on a suitable intertwining operator. Consider the monomial

$$\psi(y) = y_1 \cdots y_n$$

It is trivial to check that

- ψ in M-invariant
- ψ has weight $-\mu_n = \gamma_1 + \cdots \gamma_n$
- viewed as a polynomial in $S(\mathfrak{g})$, ψ is annihilated by $\overline{\mathfrak{n}}$.

and so ψ is the lowest weight vector of an irreducible finite-dimensional spherical representation of $\mathfrak{g}$.

■′ The highest weight vector ψ should provide the same functionality as the Kostant-Sahi Capelli operators, and so should provide a replication of circumstance (v).

Remark 3.8. The gap between "should" and "will" in the above lies in the fact that I still need to detail the transport of ψ to a K-invariant differential operator D on $K/(K \cap L)$ and then determine the eigenvalues D on the K-types in V.

Appendix A. Closure Relations for Spherical Nilpotent Orbits of Classical Real Linear Groups

To indicate exactly which spherical orbits are constructible by our sequences of strongly orthogonal noncompact weights, we display below the closure relations for the spherical orbits of classical real linear Lie groups; or rather those cases for which we've identified a nice pattern (the closure diagrams of $SU(p,q)$ and $SO(p,q)$ get rather complicated as p and q increase). The double lines in the diagram indicate the simple chains of spherical nilpotent orbits closures corresponding to sequences of strongly orthogonal noncompact weights (cf. Remark 2.1.3.). In the Hermitian symmetric cases we indicate both the chains lying in $\mathfrak{p}_+$ and those lying in $\mathfrak{p}_-$. Our notation for the orbits is somewhere between that of King and Djokovic. Briefly, following King, we indicate particular orbits by expressions of the form $(\pm n_1)^{m_1}(\pm n_2)^{m_2}\cdots(\pm n_k)^{m_k}$, where a factor of the form $(\pm n_i)^{m_i}$ indicates the occurance of a signed a row of alternating '+' and '-' signs, of length n_i, beginning with a $\pm$ sign, and occuring with multiplicity m_i. However, Djokovic's algorithm makes use of unsigned rows (actually, unsigned "genes") rather than rows that are more commonly represented as even signed rows; we indicate such an unsigned row of length n occuring with multiplicity m by a factor of the form $(n)^m$. Thus, for example,

$$(+3)^2(2)(+1)^2(-1) \quad \sim \quad \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & - & + \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} + & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} - & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

A.1. $SL(n,\mathbb{R})$.

$$\begin{tikzcd} \mathcal{O}^{I}_{(2)^{[n/2]}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(2)^{[n/2]}} \arrow[dl, Rightarrow] \\ & \mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] & \\ & \mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] & \\ & \vdots \arrow[d, Rightarrow] & \\ & \mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] & \\ & \mathcal{O}_{(1)^n} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{(2)^{[n/2]}} \arrow[d, Rightarrow] \\ \mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] \\ \mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] \\ \vdots \arrow[d, Rightarrow] \\ \mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] \\ \mathcal{O}_{(1)^n} \end{tikzcd}$$

n even $\qquad\qquad$ n odd

A.2. $SU\left(2,q\right)$.

$$\begin{tikzcd}
 & & \mathcal{O}_{(-3)^2(-1)^{q-4}} \arrow[dll] \arrow[drr] & & \\
\mathcal{O}_{(-3)(-2)(-1)^{q-3}} \arrow[dd] \arrow[drr] & & & & \mathcal{O}_{(-3)(+2)(-1)^{q-3}} \arrow[dll] \arrow[dd] \\
 & & \mathcal{O}_{(-3)(+1)(-1)^{q-1}} \arrow[d] & \mathcal{O}_{(+3)(-1)^{q-1}} \arrow[dl] & \\
\mathcal{O}_{(-2)^2(-1)^{q-2}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(+2)(-2)(-1)^{q-2}} \arrow[dl] \arrow[dr] & & \mathcal{O}_{(+2)^2(-1)^{q-2}} \arrow[dl, Rightarrow] \\
 & \mathcal{O}_{(-2)(+1)(-1)^{q-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(+2)(+1)(-1)^{q-1}} \arrow[dl, Rightarrow] & \\
 & & \mathcal{O}_{(+1)^2(-1)^{q}} & &
\end{tikzcd}$$

A.3. $SL\left(n,\mathbb{H}\right)$.

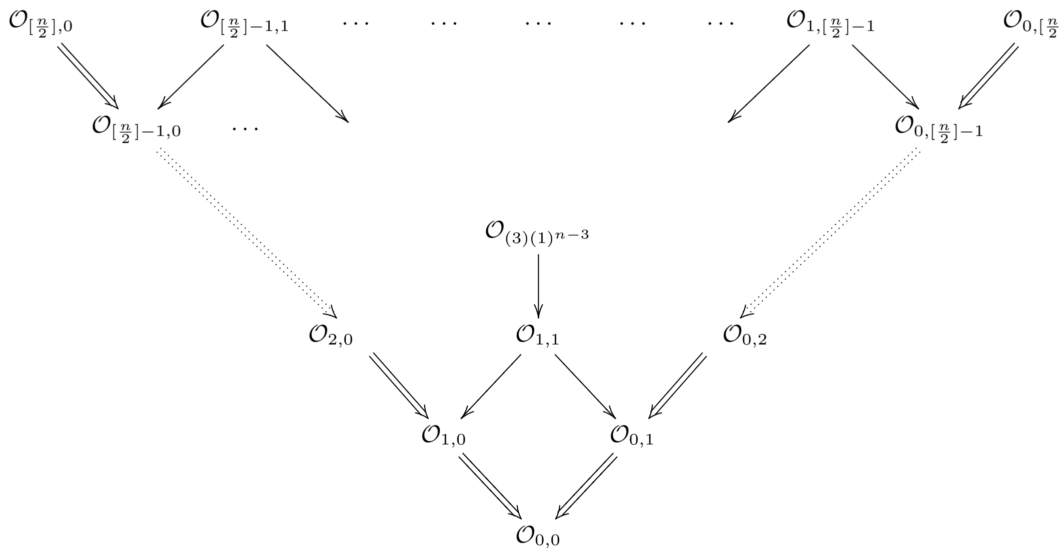
$$\begin{tikzcd}
\mathcal{O}_{(2)^{[n/2]}} \arrow[d, Rightarrow] & \mathcal{O}_{(2)^{[n/2]}(1)} \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] & \mathcal{O}_{(2)^{[n/2]-1}(1)^3} \arrow[d, Rightarrow] \\
\vdots \arrow[d, Rightarrow] & \vdots \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] & \mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] \\
\mathcal{O}_{(1)^n} & \mathcal{O}_{(1)^n}
\end{tikzcd}$$

n even $\qquad\qquad$ n odd

A.4. $SO\left(2,p\right)$; $p>4$.

$$\begin{tikzcd}
 & & \mathcal{O}_{(-3)^2(-1)^{p-4}} \arrow[d] & & \\
\mathcal{O}^{I}_{(+3)(-1)^{p-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(-3)(+1)(-1)^{p-2}} \arrow[dl] \arrow[dr] & & \mathcal{O}^{II}_{(+3)(-1)^{p-1}} \arrow[dl, Rightarrow] \\
 & \mathcal{O}^{I}_{(+2)(-2)(-1)^{p-2}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(+2)(-2)(-1)^{p-2}} \arrow[dl, Rightarrow] & \\
 & & \mathcal{O}_{(+1)^2(-1)^{p}} & &
\end{tikzcd}$$

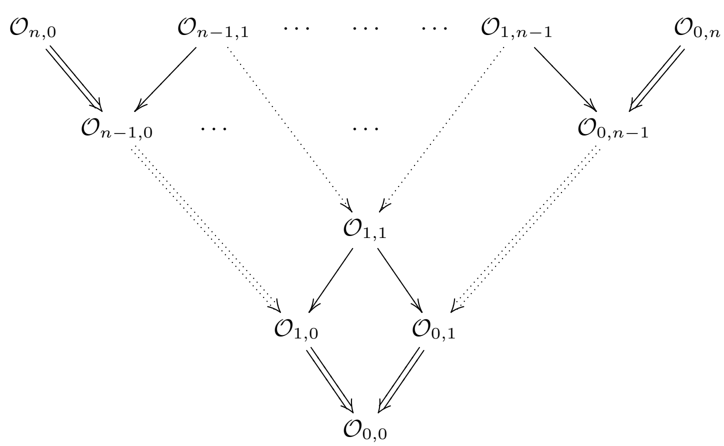
A.5. $SO^*(2n)$.



where

$$\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^r(-2)^s(1)^{n-2r-2s}}$$

A.6. $Sp(n,\mathbb{R})$.



where

$$\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^r(-2)^s(+1)^{n-r-s}(-1)^{n-r-s}}$$

A.7. $Sp(p,q) \quad p \leq q$.

$$
\begin{tikzcd}
& \mathcal{O}_{(+2)^p} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(+2)^{p-1}(+1)(-1)} \arrow[d, Rightarrow, dotted] & \\
\mathcal{O}_{(+3)(+2)(+1)^{p-3}(-1)^{q-2}} \arrow[d] \arrow[dr] & \mathcal{O}_{(+2)^3(+1)^{p-3}(-1)^{q-3}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+2)(+1)^{p-2}(-1)^{q-3}} \arrow[d] \arrow[dl] \\
\mathcal{O}_{(+3)(+1)^{p-2}(-1)^{q-1}} \arrow[dr] & \mathcal{O}_{(+2)^2(+1)^{p-2}(-1)^{q-2}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+1)^{p-1}(-1)^{q-2}} \arrow[dl] \\
& \mathcal{O}_{(+2)(+1)^{p-1}(-1)^{q-1}} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(+1)^p(-1)^q} &
\end{tikzcd}
$$