Variations on a Formula of Barbasch and Vogan

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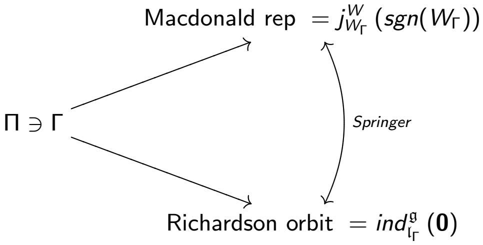
Notation

- $\mathfrak{g}$: a complex semisimple Lie algebra
- $\mathfrak{h}$: a CSA of $\mathfrak{g}$
- $\Delta \subset \Pi$: roots and simple roots
- $\mathfrak{g}^\vee$: Lie algebra dual to $\mathfrak{g}$ (e.g., $\mathfrak{sp}(2n,\mathbb{C})^\vee = \mathfrak{so}(2n+1,\mathbb{C})$)
- $\mathcal{N}_\mathfrak{g}$: set of nilpotent orbits of $G = Ad(\mathfrak{g})$ in $\mathfrak{g}$ (a finite set partially ordered via inclusion of closures).
- $\mathcal{S}_\mathfrak{g}$: the set of special nilpotent orbits (unique dense orbits in associated varieties of primitive ideals of regular integral infinitesimal character)
- W : the Weyl group of $\mathfrak{g}$ (and $\mathfrak{g}^\vee$).

Goal: Relate nilpotent orbits and Weyl group reps via common combinatorial parameters.

Paradigm



Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.

There are two basic ways of attaching to the datum $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ a nilpotent orbit in $\mathfrak{g}$.

Inclusion of Nilpotent Orbits

$$inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = G \cdot \mathcal{O}_{\mathfrak{l}} = \{X \in \mathfrak{g} \mid X = g \cdot x \quad \text{for some } g \in G \text{ , } x \in \mathcal{O}_{\mathfrak{l}}\}$$

Induction of Nilpotent Orbits Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be any extension of $\mathfrak{l}$ to a parabolic subalgebra of $\mathfrak{g}$.

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \text{unique dense orbit in } G \cdot (\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$$

Def. A nilpotent orbit is *distinguished* if it is does not meet any proper Levi subalgebra.

Theorem

(Bala-Carter) $\mathcal{N}_{\mathfrak{g}}$ is in a 1:1 correspondence with G-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}}$ is a distinguished orbit in $\mathfrak{l}$.

Parameterizing Conjugacy Classes of Levis

Fact:

$$G\text{-conjugacy classes of Levis} \quad \xleftrightarrow{1:1} \quad 2^{\Pi}/W$$

Let $\Gamma \subset \Pi$ and set

$$W_\Gamma = \langle s_\alpha \rangle_{\alpha \in \Gamma} \subset W$$
$$\Delta_\Gamma = W_\Gamma \cdot \Gamma$$
$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \Delta_\Gamma} \mathfrak{g}_\alpha$$

Let $\gamma \subset \Gamma$ such that

$$\#\Delta_\gamma + \#\Gamma = \#\left\{\alpha \in \Delta_\Gamma^+ \mid \alpha = \alpha_1 + \alpha_2 \quad ; \quad \alpha_1 \in \Delta_\gamma \ , \ \alpha_2 \in \Gamma \setminus \gamma\right\} \qquad (*)$$

Then

Fact: $ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_G}(\mathbf{0})$ is a distinguished orbit in $\mathfrak{l}_\Gamma$, and all distinguished orbits arise in this fashion.

Definition

Let Γ be any set of simple roots (a linearly indep. and mutually obtuse set). A subset $\gamma \subset \Gamma$ will called **distinguished** if (*) is satisfied.

Combinatorial Bala-Carter

$$\mathcal{N}_{\mathfrak{g}} \quad \xleftrightarrow{\ 1:1\ } \quad \{(\Gamma,\gamma) \mid \gamma \subset \Gamma \subset \Pi \quad \text{satisfying } (*)\} \, / W$$

$$\mathcal{O}_{(\Gamma,\gamma)} \equiv inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} \left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma} (\mathbf{0}) \right)$$

Set

$$\mathcal{BC}_{\mathfrak{g}} = \{(\Gamma,\gamma) \mid \gamma \subset \Gamma \subset \Pi \quad \text{satisfying } (*)\} \, / W$$

Orbit Duality for $\mathfrak{sl}_n$

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Then

$$\mathcal{N}_{\mathfrak{g}} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{partitions of } n\}$$

$$\mathbf{p} \longmapsto \mathcal{O}_{\mathbf{p}} = \text{orbit of} \begin{pmatrix} J_{p_1} & 0 & \cdots & 0 \\ 0 & J_{p_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & J_{p_k} \end{pmatrix} \quad , \quad J_{p_i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Theorem

(Gerstenhaber) The partition transpose map $t : \mathcal{P}(n) \longrightarrow \mathcal{P}(n)$ induces an order reversing involution

$$d : \mathcal{N}_{\mathfrak{sl}_n} \longrightarrow \mathcal{N}_{\mathfrak{sl}_n} : \quad \mathcal{O}_{\mathbf{p}} \longrightarrow \mathcal{O}_{\mathbf{p}^t}$$

on the set of nilpotent orbits of $\mathfrak{sl}(n, \mathbb{C})$.

Spaltenstein Duality

Theorem

(Spaltenstein) Let $\mathfrak{g}$ be a simple Lie algebra. Then there is a unique map $d : \mathcal{N}_{\mathfrak{g}} \longrightarrow \mathcal{N}_{\mathfrak{g}}$ such that

- $d^2(\mathcal{O}) \leq \mathcal{O}$
- $d\left(inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{prin})\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0})$.
- $image(\mathcal{O}) =$ *special nilpotent orbits*

Consider the map $\eta_{\mathfrak{g}} : \mathcal{N}_{\mathfrak{g}} \longrightarrow \mathcal{N}_{\mathfrak{g}^\vee}$ defined by

$$\mathcal{O} \ni x \longrightarrow \{x, h, y\} \longrightarrow \frac{1}{2}h = \mu_{\mathcal{O}} \in \left(\mathfrak{h}^\vee\right)^* \longrightarrow J_{\mathcal{O}} = \max\left\{Prim\left(\mathfrak{g}^\vee\right)_{\mu_O}\right\}$$
$$\longrightarrow AssocVar\left(U\left(\mathfrak{g}^\vee\right)/J_{\mathcal{O}}\right) \xrightarrow{\text{unique dense orbit}} \eta_{\mathfrak{g}}(\mathcal{O}) \in \mathcal{N}_{\mathfrak{g}^\vee}$$

Theorem

(Barbasch-Vogan, 1985) The map $\eta_{\mathfrak{g}}$ has the following properties:

- *If $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$ then $\eta_{\mathfrak{g}}(\mathcal{O}_2) \subset \overline{\eta_{\mathfrak{g}}(\mathcal{O}_1)}$*
- $\eta_{\mathfrak{g}} \circ \eta_{\mathfrak{g}^\vee} \circ \eta_{\mathfrak{g}} = \eta_{\mathfrak{g}}$
- *Image* $(\eta_{\mathfrak{g}}) = \{$*special nilpotent orbits in* $\mathfrak{g}^\vee\}$

Theorem

(Barbasch-Vogan) If $\mathcal{O}_{\mathfrak{l}^\vee} \in \mathcal{N}_{\mathfrak{l}^\vee}$ *is an orbit in a Levi subalgebra* $\mathfrak{l}^\vee$ *of* $\mathfrak{g}^\vee$*, then*

$$\eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}\left(\mathcal{O}_{\mathfrak{l}^\vee}\right)\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\eta_{\mathfrak{l}^\vee}\left(\mathcal{O}_{\mathfrak{l}^\vee}\right)\right)$$

Variation 1

Let

$$\mathcal{BC}_{\mathfrak{g}^\vee} = \left\{ \left(\Gamma^\vee, \gamma^\vee \right) \mid \gamma^\vee \subset \Gamma^\vee \subset \Pi_{\mathfrak{g}^\vee} \quad \text{satisfying } (*) \right\} / W$$

and define $\Phi : \mathcal{BC}_{\mathfrak{g}^\vee} \longrightarrow \mathcal{S}_{\mathfrak{g}}$ by

$$\begin{aligned} \Phi\left(\Gamma^\vee, \gamma^\vee\right) &= \eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}_{\Gamma^\vee}}^{\mathfrak{g}^\vee}\left(ind_{\mathfrak{l}_{\gamma^\vee}}^{\mathfrak{l}_{\Gamma^\vee}}\left(\mathbf{0}\right)\right)\right) \\ &= ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\eta_{\mathfrak{l}_{\Gamma^\vee}}\left(ind_{\mathfrak{l}_{\gamma^\vee}}^{\mathfrak{l}_{\Gamma^\vee}}\left(\mathbf{0}\right)\right)\right) \\ &= ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}_\gamma, prin}\right) \end{aligned}$$

$\Longrightarrow$ An orbit-intrinsic characterization of special orbits (no reference to primitive ideals or special representations of Weyl groups)

N.B. use of dual parameters

Variation 2

Weyl group analogs

$$\begin{array}{rcll} \mathcal{O}_{prin} & \longleftrightarrow & \mathbf{1}_W & \\ \mathbf{0}_{\mathfrak{g}} & \longleftrightarrow & sgn\,(W) & \\ ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\,() & \longleftrightarrow & j_{W_\Gamma}^{W}\,() & \text{(truncated induction)} \\ \eta_{\mathfrak{g}} & \longleftrightarrow & \varepsilon_W & \left(\text{Lusztig's involution of } \widehat{W} \text{ w/ twist by } sgn(W)\right) \end{array}$$

$$\begin{array}{l} \Phi \quad : \quad \mathcal{BC}_{\mathfrak{g}^\vee} \longrightarrow \mathcal{S}_{\mathfrak{g}} \quad ; \quad (\Gamma^\vee, \gamma^\vee) \longrightarrow \eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}_{\Gamma^\vee}}^{\mathfrak{g}^\vee}\left(ind_{\mathfrak{l}_{\gamma^\vee}}^{\mathfrak{l}_{\Gamma^\vee}}(\mathbf{0})\right)\right) \\ \downarrow \\ \Psi \quad : \quad \mathcal{BC}_{\mathfrak{g}^\vee} \longrightarrow \widehat{W}_{spec} \quad : \quad (\Gamma^\vee, \gamma^\vee) \longrightarrow j_{W_\Gamma}^{W}\left(\varepsilon_{W_{\Gamma^\vee}}\left(j_{W_{\gamma^\vee}}^{W_{\Gamma^\vee}}\left(sgn\left(W_{\gamma^\vee}\right)\right)\right)\right) \end{array}$$

$\Longrightarrow$ an alternative W-intrinsic characterization of special representations (no generic degree polynomials required).

Variation 3

Let

$$\Pi_e = \Pi \cup \{\text{lowest root}\}$$

Set

$$\mathcal{BC}_{e,\mathfrak{g}} = \{(\Gamma, \gamma) \mid \Gamma \subset \Pi_e \;,\; \gamma \subset \Gamma \;\; \text{satisfying (*)}\}$$

Theorem

$$\widetilde{\Psi} : \mathcal{BC}_{e,\mathfrak{g}^\vee} \longrightarrow \widehat{W} : (\Gamma^\vee, \gamma^\vee) \longrightarrow j_{W_\Gamma}^{W}\left(\varepsilon_{W_{\Gamma^\vee}}\left(j_{W_{\gamma^\vee}}^{W_{\Gamma^\vee}}\left(sgn\left(W_{\gamma^\vee}\right)\right)\right)\right)$$

maps $\mathcal{BC}_{e,\mathfrak{g}^\vee}$ *onto* $\widehat{W}_{orbit}$*, where*

$$\widehat{W}_{orbit} = \left\{\sigma \in \widehat{W} \mid \sigma \sim \left(\mathcal{O}, \mathbf{1}_{A(\mathcal{O})}\right)\right\}$$

$\Longrightarrow$ a W-intrinsic characterization of Springer representations.