Representation-Theoretical Interpretation of Interaction Part I, Massless UIRs

1. Introduction

The title of these talks alludes to a seminal paper by Ryoyu Utiyama, in which the four fundamental forces of nature – the electromagnetic force, the strong and weak nuclear forces, and the gravitational force – are uniformly presented as consequences of local gauge invariance. In brief, Utiyama begins with the Lagrangian field theories of free (non-interacting) fields corresponding to the kinematical content of the theories and invariant under a Lie group G. Interactions between particles are then introduced by replacing the ordinary derivatives $\frac{\partial}{\partial x_\mu}$in the free Lagrangians by covariant derivatives. In so doing, new terms are added to which entend the global symmetries of the orginal Lagrangian to local *gauge* symmetries and simultaneously introduce coupling terms between the matter fields and the force fields.

For example, in the case of QED (where electron and photon fields interact), the free field Lagrangian is

(1) $$\mathcal{L}_{free}=\int_{\mathbb{R}^{3,1}} d^4x\ \overline{\psi}\left(x\right)\gamma_\mu\partial^\mu\psi+\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right)$$

and it is invariant under global phase transformations where $\psi\left(x\right)\to e^{i\phi}\psi\left(x\right)$. To extend this invariance to local (x-dependent) phase transformations $\psi\left(x\right)\to e^{i\phi(x)}\psi\left(x\right)$, one replaces the derivatives with covariant derivatives

(2) $$\frac{\partial}{\partial_\mu}\to\nabla_\mu=\frac{\partial}{\partial_\mu}+ieA_\mu\qquad,\qquad\overline{\psi}\left(x\right)\partial_\mu\gamma^\mu\psi\left(x\right)\to\overline{\psi}\left(x\right)\nabla_\mu\psi\left(x\right)=\overline{\psi}\left(x\right)\partial_\mu\gamma^\mu\psi\left(x\right)+ie\overline{\psi}\left(x\right)\gamma_\mu A^\mu\psi\left(x\right)$$

and this, in turn, leads to the following interaction term beween the electron and photon fields

(3) $$I_{QED}=ie\int_{\mathbb{R}^{3,1}} d^4x\overline{\psi}\left(x\right)\gamma_\mu A^\mu\psi\left(x\right)$$

Thus, from Utiyama's point of view, the interaction between electrons and photons arise as a consequence of enhancing the global $U\left(1\right)$ phase symmetry of the non-interacting theory to be a local $U\left(1\right)$ symmetry. As pretty as it is to relate interactions to symmetries in this way, it is really just a design principle for producing gauge invariant *classical* field theories (which in turn must quantized, renormalized, etc. before actually making direct contact with experiment).

That said, the expression (3) is nevertheless satisfying in another way, as it seems to say that total electromagnetic interaction has something to do with the likelihood of a electron, an anti-electron and a photon all being at the same point x in space-time and then summing over all x.

However, representation-theoretically, this expression has another interpretation. To set this up, we first think of elementary particles as corresponding to particular unitary irreducible representation of the underlying spacetime symmetry group (the Poincaré group), and then regard the fields $\psi\left(x\right)$, $A^\mu\left(x\right)$, ... as representing particular Hilbert space elements of the UIRs (unitary irreducible representations) of the Poincaré group (the symmetry group of space-time),

$$\psi\in\mathcal{H}_{electron}\quad,\quad\overline{\psi}\in\mathcal{H}_{electron}^{\dagger}\quad,\quad A\in\mathcal{H}_{photon}$$

We can then view the interaction term (1) as a trilinear form on $\mathcal{H}^{\dagger}_{electron} \otimes \mathcal{H}_{photon} \otimes \mathcal{H}_{electron}$ (conjugate-linear on the first factor) which is akin to the familar trilinear of linear algebra

$$\mathbf{v}^{\dagger}\mathbf{M}\mathbf{v} \quad , \quad \mathbf{v} \in \mathbb{C}^n \quad , \quad \mathbf{M} \text{ an } n \times n \text{ matrix}$$

A special case of which is when $\mathbf{M} \in GL_n(\mathbb{C})$ and then such a form would yield an $GL_n(\mathbb{C})$-invariant trilinear form on

$$(\text{dual fundamental representation}) \otimes (\text{adjoint representation}) \otimes (\text{fundamental representation})$$

and then this in turn generalizes to a trilinear form corresponding to the matrix elements of a representation of $GL_n(\mathbb{C})$:

$$\left\langle \psi_i, \pi(X)\,\psi_j \right\rangle \quad , \quad X \in \mathfrak{g}$$

which in turn can be viewed, in a direct and rigorous representation-theoretical, quantum-mechanical way (in the manner of Dirac, von Neumann, Wigner, Mackey, et al.), as

$$\left\langle \psi_i | \pi(X) | \psi_j \right\rangle$$

corresponding to the probability of a state ψ_j transitioning to a state ψ_i after measuring an observable X.

It is the last point of view that we shall pursue in this talk.

2. Our General Setting

Rather than force and matter fields, the fundamental objects in this talk are to be *elementary quantum systems* in the sense of Newton-Wigner. That is to say, an *elementary quantum mechanical system* is a quantum mechanical system that carries an unitary irreducible representation (hereafter, a UIR) $(\pi, \mathcal{H})$ of a symmetry group G. Interpreted quantum mechanically;

physical states	$\longleftrightarrow$	$\psi \in \mathcal{H}$
observables	$\hookleftarrow$	$X \in \mathfrak{g} = Lie(G)$
probablistic interpretation	$\longleftrightarrow$	G-invariant hermitan form on $\mathcal{H}$
indivisable nature of system	$\longleftrightarrow$	irreducible representation

This point of view will allow us to be a little more agnostic about the nature of underlying spacetime, while still providing a firm computational setup for computing the S-matrix (which describes particle scattering amplitudes).

We note at this juncture that a seemingly disparate, yet key example of an elementary quantum system, is the hydrogen atom. First of all, the full group of dynamical symmetries of the H-atom is actually $SO(4,2)$, a group which is also naturally associated with massless particles). Secondly, although the exact UIR to which the its physical states belong is somewhat undecided (there seems to be a disagreement on the correct eigenvalue for the 4^{th} order Casimir operator), the picture of the state space of the H-atom as a hierarchy of energy levels, whose degeneracies are split by their weights with respect to a compact subgroup ($SO(3)$), provides a much closer analog of the $(\mathfrak{g}, K)$-module picture of UIRs that modern representation theorists have developed.

Our bold hope is that, starting with just the UIRs of the Standard Model (that is to say, its bare kinematical content) , one can not only construct its S-matrix, but also, by simply not distinguishing the spacetime symmetries from the internal $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$ symmetries (which to the electromagnetic, strong and weak forces), may account for gravitational effects as well (as local Poincaré invariance is built in from the ground up.) Indeed, we imagine spacetime not as a primordial object, but rather the result of patching together local systems of imprimitivity associated with the UIRs of the theory - thus, the localizability of UIRs can be viewed as the *sine qua non* for both spacetime and local interactions.[1]

[1]Reference: Newton-Wigner, Wightman, Angelopoulos-Flato.

In these talks, however, we shall try to constrain our imagination and restrict our attention to the massless electron and photon sector of the Standard Model with the goal of constructing an S-matrix for massless QED directly from the massless electron and photon UIRs.

In this first talk, I'll introduce the family of massless representations of the Poincaré group, describe their unique extensions to conformal group(s), and point out some very special properties these representations have when considered as representations of a real reductive group. I'll then provide three nice realizations of the massless representations:

- Field Theoretical construction
- Unitary Highest Weight Module construction
- Quantum Mechanical construction via a system of quantum harmonic oscillators

Each of these realizations sheds a different light as to the nature of massless particles as physical systems.

The second talk will be concerned with the problem of introducing interactions between massless particles. We will focus on the case of QED where massless electrons and photons interact. Here, rather than Utiyama's "local-guage-invariance/minimal-substitution" method, we'll look for a purely representation-theoretical explanation of the fundamental Feynman diagrams of the theory. In so doing, we'll also develop a purely representation-theoretical/quantum-mechanical means of computing Feynman diagrams and S-matrix elements.

It should be clear that the ideas developed here for understanding QED can be immediately generalized to develop a representation-theoretical construction of the Standard Model.

3. Massless UIRs of the Poincare Group

Let

$$\{T_\mu \mid \mu = 0,1,2,3\} \cup \{L_{\mu\nu} \mid 0 \leq \mu < \nu \leq 3\}$$

be the usual basis for the Lie algebra $\mathfrak{g}_{\mathcal{P}}$ of the Poincaré group, and let $g^{\mu\nu} = diag\,(1,-1,-1,-1)$ be the usual Minkowski metric tensor. We have

$$\begin{aligned} [T_\mu, T_\nu] &= 0 \\ [L_{\mu\nu}, T_\lambda] &= g_{\nu\lambda}T_\mu - g_{\mu\lambda}T_\nu \\ [L_{\mu\nu}, L_{\lambda\rho}] &= g_{\nu\lambda}L_{\mu\rho} - g_{\mu\lambda}L_{\nu\rho} - g_{\nu\rho}L_{\mu\lambda} + g_{\mu\rho}L_{n\lambda} \end{aligned}$$

The (non-trivial) center of $U\left(\mathfrak{g}_{\mathcal{P}}\right)$ is generated by the two Casimir elements

$$T^2 = T_\mu T^\mu \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g^{\mu\nu} T_\mu T_\nu \quad \text{(the mass-squared operator)}$$

and

$$W^2 = W_\mu W^\mu$$

where

$$W_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} M^{\nu\lambda} T^\rho \quad \text{(the Pauli-Lubanski vector)}$$

A UIR (π, V) of the Poincaré group is called *massless* if the element T^2 vanishes identically on V. It turns out that the eigenvalues of energy operator T_0 on such a representation are either always positive, always negative, or 0. In the first (resp. second) case, we call the representation, a positive (resp. negative) energy representation. The case where $T_0 = 0$ corresponds to the trivial representation of $\mathcal{P}$ which we'll henceforth exclude from the designation *massless representation.*[2]

[2]We also exclude the massless, continuous (a.k.a. infinite) spin representations of $\mathcal{P}$ as they have yet to be observed in nature.

For the massless representations so defined, it turns out that

$$W_\mu P^\mu = 0$$

as well, and one, in fact has $W_\mu = hP_\mu$. The proportionality constant h provides another invariant that splits of the family of massless representations. It is called the *helicity* or *handedness* of the massless representation and its value on a massless representation is always half-integer. Thus, we have three families of massless UIRs of $\mathcal{P}$

- $U_{0,\pm}$: the scalar massless representations;characterized by $P^2 = 0$ and $h = 0$, $sign(T_0) = \pm$
- $U_{L,s,\pm}$, $s = \frac{1}{2}, 1, \frac{3}{2} \ldots$: the left-handed massless representations with spin s: characterized by $P^2 = 0$, $h = s$, $sign(T_0) = \pm$
- $U_{R,s,\pm}$, $s = \frac{1}{2}, 1, \frac{3}{2} \ldots$: the right-handed massless representations with spin s: characterized by $P^2 = 0$ and $h = -s$, $sign(T_0) = \pm$

4. Massless Representations of $SO(4,2)$

A very special property of the massless representations of $\mathcal{P}$ is that they each have unique extension to an irreducible unitary representation of the *conformal group* $SO(4,2)$. Thus, corresponding to each massless representation π of $\mathcal{P}$ there is a unique (up to equivalence) UIR $\widetilde{\pi}$ of $SO(4,2)$ such that $\widetilde{\pi}|_{\mathcal{P}} = \pi$. This is an especially nice feature since it allows us to replace the somewhat cumbersome Poincaré group (a semidirect product group) with a (real form of a) simple classical Lie group with no loss of content: purely Poincaré results can recovered simply by restriction. Moreover, as we shall see below, examining massless representations in the $SO(4,2)$ setting provides some new insights and new tools for analysis.

As a basis for the Lie algebra $\mathfrak{g} = \mathfrak{so}(4,2)$, we adopt generators $\{L_{ab} = -L_{ba} :| \, 0 \leq a < b \leq 5\}$ satisfying

$$[L_{ab}, L_{cd}] = \eta_{bc}L_{ad} - \eta_{bd}L_{ac} - \eta_{ac}L_{bd} + \eta_{ad}L_{bc} \qquad , \qquad \eta = diag(1,-1,-1,-1,-1,1)$$

and for which the generators of a Poincaré subalgebra embed nicely

$$\begin{aligned} T_\mu &= L_{a4} + L_{a5} \quad , \quad a = \mu \in \{0,1,2,3\} \\ L_{\mu\nu} &= L_{ab} \qquad 0 \leq \mu = a < \nu = b \leq 5 \end{aligned}$$

When viewed purely as representations of $\mathfrak{g}$ $(= \mathfrak{so}(4,2))$ the massless representations are rather special.

(i) They are unitary highest weight modules. In fact, each appears at a first reduction point in Enright-Howe-Wallach's classification of unitary highest weight modules.
(ii) They are *singular unitary representations* in the sense of [Vogan, 2006]. Thus, their K-types are multiplicity-free and the highest weights of these K-types lie along a single line in $\mathfrak{t}^*$ ($\mathfrak{t}$ being a compact Cartan subalgebra for both $\mathfrak{k}$ and $\mathfrak{g}$). In fact, the weights of a massless representation (with respect to the compact Cartan subalgebra $\mathfrak{t}$) are multiplicity-free.
(iii) The mass-squared operator $T_\mu T^\mu$ is no longer an invariant operator, yet it must still vanish identitically on the massless representations. Because of this, the entire ideal generated by $T_\mu T^\mu$ in $U(\mathfrak{g})$ must vanish on the massless representations. This hints at the existence of a large ideal in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, which vanishes on each physical state vector. Indeed, if $(\pi_h, \mathcal{H}_h)$ is a massless UIR of helicity h, the *annihilator* of π_h

$$Ann(\pi_h) \equiv \{X \in U(\mathfrak{g}) \mid Xv = 0 \ , \ \forall \ v \in \mathcal{H}_h\}$$

is a maximal proper 2-sided ideal in $U(\mathfrak{g})$. This ideal is generated by second order elements[3]

$$\frac{1}{2}L_{ab}L^{ab}+\frac{h^2}{4}-3 \quad \sim \quad F_{[0,0,0]}$$

$$\left\{\eta^{ac}L_{ab}L_{cd}+\eta^{ac}L_{ad}L_{cb}-\frac{1}{3}\eta_{bd}L_{ad}L^{ad} \mid b,c\in\{0,1,2,3,4,5\}\right\} \quad \sim \quad F_{[2,0,0]}$$

$$\left\{\varepsilon^{abcdef}L_{cd}L_{ef}+8hL^{ab} \mid a,b\in\{0,1,2,3,4,5\}\right\} \quad \sim \quad F_{[1,0,1]}$$

where on the right we specify the irreducible finite-dimensional representation of $\mathfrak{g}\approx D_3$, in terms of their components w.r.t. a basis of fundamental weights, corresponding to the generators on the left.

(iv) The associated variety of their annihilators is the minimal (non-trivial) nilpotent coadjoint orbit in $\mathfrak{g}^*$.

5. The K-types and Weights of Massless Representations

Suppose π is an irreducible unitary representation of a simply-connected, connected, simple Lie group G, and let K be a maximal compact subgroup of G. Restricting the representation π to K, we get a decomposition

$$\pi|_K=\bigoplus_{\Lambda}m_\Lambda F_\Lambda$$

where each F_Λ is an irreducible finite-dimensional representation of K and the corresponding coefficient m_Λ the (finite) multiplicity of F_Λ in $\pi|_K$. The isotypical components F_Λ with non-zero multiplicities are referred to as the *K-types* of π. Since K is connected and compact, each F_Λ is finite-dimensional and unitary, and determined by its highest weight (which we can take to be the meaning of the label Λ).

In the case at hand, we have $\mathfrak{g}=\mathfrak{so}(4,2)$, $\mathfrak{k}_0=u(1)\oplus\mathfrak{su}(2)_L\oplus\mathfrak{su}(2)_R$ (the subscripts L and R having to do with the left-handed-ness and right-handed-ness of the massless UIRs). As remarked above, the massless representations of $\mathfrak{so}(4,2)$ are *singular unitary representations* in the sense of Vogan. As such, the (highest weights of) their K-types are multiplicity-free and lie along a single ray in $\mathfrak{t}^*$

$$\pi|_K=\bigoplus_{n=0}^{\infty}F_{\Lambda_0+n\beta}$$

Here $\beta=[-1,1,1]$ is the highest weight of the representation of K on $\mathfrak{p}_-$ and Λ_0 is the highest weight of representation. Thus,

$$\pi|_K=\begin{cases}\bigoplus_{n=0}^{\infty}F_{\left[-1-\frac{s}{2}-n,s+n,n\right]} & \text{when } \pi \text{ is a left-handed, helicity } s, \text{ massless representation}\\ \bigoplus_{n=0}^{\infty}F_{\left[-1-\frac{s}{2}-n,n,s+n\right]} & \text{when } \pi \text{ is a left-handed, helicity } -s, \text{ massless representation}\end{cases} \tag{1}$$

It's well known that an irreducible finite dimensional representation (σ,V) of $\mathfrak{su}(2)$ decomposes into 1-dimensional weight spaces

$$V=\bigoplus_{i=0}^{n}V_{n-2i} \quad ; \quad n=\dim(V)-1 \tag{2}$$

From (1) and (2), we can quickly deduce that the weights of a massless representation π are

$$W_\pi=\begin{cases}\left\{\left[-1-\frac{s}{2}-n,s+n-2i,n-2j\right] \mid n\in\mathbb{Z}_{\geq 0}\ ;\ i=0,...,s+n\ ;\ j=0,\dots,n\right\} & ; \quad \pi \text{ is massless rep with helicity } s\\ \left\{\left[-1-\frac{s}{2}-n,s-2i,n+s-2j\right] \mid n\in\mathbb{Z}_{\geq 0}\ ;\ i=0,...,n\ ;\ j=0,\dots,n+s\right\} & ; \quad \pi \text{ is a massless rep with helicity}\end{cases}$$

Moreover, the corresponding weight spaces are 1-dimensional; and so to each weight $w\in W_\pi$, there corresponds is a unique physical state.

[3][AFFS,1981]

6. Three Realizations of the Massless Representations of $SO(4,2)$

6.1. Massless Representations as Conformal Fields. Although part of the point of this paper is to not rely on field theoretical constructs, the field theoretical realizations of massless conformal fields are important both historically and conceptually. This realization goes back to a 1936 paper by P.A.M. Dirac (Ann. Math. **37**,429-442, 1936), and is the foundation of many studies of conformally invariant field theories (citations like Mack-Salam, BFH, etc).

This construction begins with fields (i.e. sections of some fiber bundle) on $\mathbb{R}^{2,4}$ upon which $SO(2,4)$ acts naturally as a pseudo-rotation group; but the main goal is to restrict fields and field equations on $\mathbb{R}^{2,4}$ to the projective cone $y^2 = 0$, which is isomorphic to a particular compactification of Minkowski space (thus, establishing $\mathfrak{so}(2,4)$ covariant field theories on Minkowski space.) To achieve this goal, however, it turns out that the original fields on $\mathbb{R}^{2,4}$ have to be homogeneous functions of the coordinates of a particular degree (so that they correspond to sections of a half-density bundle), and so that the wave operators that single out particular massless UIRs have well-defined restrictions to the projective cone.

For massless UIRs with spin ≤ 1, these realizations look particularly natural to physicists. For example,

- The spin 0 massless UIRs corresponds to scalar fields that are homogeneous of degree -1 satisfying
$$\partial_a \partial^a \phi(y) = 0$$
- The spin $\frac{1}{2}$ massless UIRs correspond to (8-component conformal-) spinor fields $\chi(y)$ that are homogeneous of degree -2, satisfying
$$-2i\left(\gamma^{ab} L_{ab} + 2\right)\chi(y) = 0$$
where $L_{ab} = y_a \partial_b - y_b \partial_a$ and the γ^{ab} are the generators of the spinor representation of $\mathfrak{so}(4,2)$ formed from the $SO(2,4)$ Clifford algebra:
$$\left\{\beta^a, \beta^b\right\} = \eta^{ab} I \quad \text{and } \gamma^{ab} = \frac{i}{4}\left[\beta^a, \beta^b\right]$$
- The spin 1 massless UIRs correspond to (6-component) vector fields $A_a(y)$ that are homogeneous of degree -1 and satisfy
$$\begin{aligned} \partial_a \partial^a A_b(y) &= 0 \quad \text{(conformal wave equation)} \\ \partial^a A_a(y) &= 0 \quad \text{(conformal Lorentz condition)} \\ y^a A_a(y) &= 0 \quad \text{(conformal subsidiary condition)} \end{aligned}$$

What's especially nice about the last two realizations is that they can be derived, via an action principle, from an invariant Lagrange function. Moreover, the electromagnetic interaction can be sucessfully introduced by generalizing Utiyama's method of derivative substitution to
$$L_{ab} \to L_{ab} + e\left(y_a A_b - y_b A_a\right)$$
When this substitution is carried out on the free Lagrangians, the Lagrangian gains an interaction term of the form
$$I_{QED} = \frac{1}{2}\int_{\mathcal{C}} dy \overline{\chi}(y)\, \beta^a \beta^b \left(y_a A_b(y) - y_b A_a(y)\right)\chi(y)$$
and total Lagrangian becomes invariant under a local gauge transformations. Note that, just like ordinary (Minkowski space) QED, the interaction term leads to invariant forms on $L_{\left[-\frac{3}{2},1,0\right]} \otimes L_{[-0,2,0]} \otimes L_{\left[-\frac{3}{2},1,0\right]}$ and $L_{\left[-\frac{3}{2},0,1\right]} \otimes L_{[-0,0,2]} \otimes L_{\left[-\frac{3}{2},0,1\right]}$ (which we regard as the left and right handed sectors of conformal QED).

We note, however, that realization of the higher spin massless UIRs are somewhat less natural. For example, the spin 2 graviton representations are **not** realizable as symmetric 2-tensor fields.

6.2. Massless Representations as Unitary Highest Weight Modules. Let $\mathfrak{g}$, $\mathfrak{k}$ be the complexifications of $\mathfrak{g}_0 \approx \mathfrak{so}(2,4)$ and its maximal compact subalgebra $\mathfrak{k}_0 = \mathfrak{u}(1) \oplus \mathfrak{so}(4) \approx \mathfrak{u}(1) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ (the subscripts L and R on the two commuting $\mathfrak{su}(2)$ subalgebras will end up being associated with, respectively, the ground states of left-handed and right-handed massless UIRs). Setting

$$\begin{aligned} P_0 &= -iL_{05} \in \mathfrak{u}(1) \\ L_0 &= -i(L_{12} + L_{34}) \in \mathfrak{su}(2)_L \\ R_0 &= -i(L_{12} - L_{34}) \in \mathfrak{su}(2)_R \end{aligned}$$

we have Cartan subalgebras for each of the commuting components $\mathfrak{u}(1)$, $\mathfrak{su}(2)_L$, $\mathfrak{su}(2)_R$ of $\mathfrak{k}$, and once we set

$$\mathfrak{t} = span_{\mathbb{C}}(P_0\ ,\ L_0\ ,\ R_0)$$

we obtain a common Cartan subalgebra for both $\mathfrak{k}$ and $\mathfrak{g}$.

Remark 1.1. For any real, noncompact, semisimple Lie algebra $\mathfrak{g}_0$, the following statements are equivalent:

Statement 1.2.
- $\mathfrak{g}_0$ *has a compact Cartan subalgebra*
- $\mathfrak{k}_0$ *has a 1-dimensional center*
- $\mathfrak{g}_0$ *is of hermitian-symmetric type; meaning* $\mathfrak{g}_0$ *is the Lie algebra of a semisimple group* G *such that whenever* K *is a maximal compact subgroup of* G*,* G/K *is a hermitian symmetric space.*
- $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ *has a decomposition*

$$\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_-$$

with $\mathfrak{k}$ *a maximal compact subalgebra, and* $\mathfrak{p}_+$ *and* $\mathfrak{p}_-$ *both abelian subalgebras.*

Remark 1.3. Furthermore, the construction we provide below extends to any semisimple Lie algebra of hermitian-symmetric type.

For any representation (π, V) of $\mathfrak{so}(4,2)$, the simultaneous eigenspaces of P_0, L_0, R_0 will be referred to as the *weight spaces* of (π, V),[4] and the corresponding triples $[t_0, \ell_0, r_0]$ of eigenvalues are called the *weights* of (π, V). Since our Cartan subalgebra is compact, the weights of a UIR of $\mathfrak{so}(4,2)$ will always be a discrete set (much like the stationary states of the H-atom). We adopt the natural lexicographic ordering of weights

$$[t_0, \ell_0, r_0] \le [t_0', \ell_0', r_0'] \quad \Rightarrow \quad t_0 < t_0' \quad \text{or } (t_0 = t_0') \textit{ and } (\ell_0 < r_0) \quad \text{or} \quad (t_0 = t_0') \textit{ and } (\ell_0 = \ell_0') \textit{ and } r_0 \le r_0'$$

The *highest weight* of a representation (π, V) is the maximal weight of (π, V) with respect to this ordering (if such a maximal weight exists).

We first note that the center $\mathfrak{z}$ of $\mathfrak{k}$ is generated by P_0, while the semisimple part of $\mathfrak{k}$ is just $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ and the way we have set things up,

$$\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$$

where

$$\begin{aligned} \mathfrak{p}_- &= \{x \in \mathfrak{g} \mid [P_0, x] = -x\} \\ \mathfrak{k} &= \{x \in \mathfrak{g} \mid [P_0, x] = 0\} \\ \mathfrak{p}_+ &= \{x \in \mathfrak{g} \mid [P_0, x] = -x\} \end{aligned}$$

Moreover,

$$\mathfrak{q} = \mathfrak{k} + \mathfrak{p}_+$$

and its opposite

$$\overline{\mathfrak{q}} = \mathfrak{k} + \mathfrak{p}_-$$

are both parabolic subalgebras of $\mathfrak{g}$.

[4] Thus, we will not be considering the weights with respect to other Cartan subalgebras.

Our ordering of weights applies in particular to the roots Δ of $\mathfrak{g}$, as well as the roots $\Delta_{\mathfrak{c}}$ of $\mathfrak{k}$. We'll say a weight $\Lambda \in \mathfrak{t}^*$ is dominant integral for Δ_c^+ if

$$\frac{2\langle\Lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{\geq 0} \quad \forall\, \alpha \in \Delta_c^+$$

By the Theorem of the Highest Weight, and the Unitary Trick, to each Δ_c^+-dominant integral weight Λ, there corresponds an irreducible, finite-dimensional unitary representation F_Λ of $\mathfrak{k}$. Such a representation readily extends to a representation of the parabolic subalgebra

$$\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}_+$$

by mandating that the noncompact radical $\mathfrak{p}_+$ acts trivally.

Definition 1.4. *Let Λ be a Δ_c^+-dominant weight and let F_Λ be the corresponding representation of $\mathfrak{q}$. The highest weight module corresponding to Λ is the generalized Verma module*

$$N_\Lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_\Lambda$$

Because $\mathfrak{p}_-$ is abelian, it is easy to see that

$$N_\Lambda \approx S(\mathfrak{p}_-) \otimes F_\Lambda \quad \text{as a } \mathfrak{k}\text{-module}$$

Thus, the elements of N_Λ can be thought of as vector-valued polynomials.

On such a module there is always a unique (up to rescaling) invariant sesquilinear form

$$\langle X \otimes \xi_1, Y \otimes \xi_2\rangle \equiv \langle \xi_1, P(\sigma(X)Y)\xi_2\rangle$$

where σ is the conjugate-linear involutive anti-automorphism of $U(\mathfrak{g})$ which equals -1 on the real form $\mathfrak{g}_0$, and $P : U(\mathfrak{g}) \to U(\mathfrak{k})$ is the natural projection corresponding to decomposition

$$U(\mathfrak{g}) = \mathfrak{p}_- U(\mathfrak{g}) \oplus U(\mathfrak{k}) \oplus U(\mathfrak{g})\,\mathfrak{p}_+$$

These modules possess a unique irreducible quotient

$$L_\Lambda = N_\Lambda / Rad(\langle\ ,\ \rangle)$$

on which the sesquilinear form is non-degenerate. Whenever the induced form is also positive-definite, $L_\Lambda$is called a *unitary highest weight module* (and of course it is also always a unitary irreducible representation).

For the Cartan subalgebra $\mathfrak{t}$ chosen above, the (positive energy) massless representations of $\mathfrak{so}(4,2)$ are precisely those unitary highest weight modules whose highest weights w.r.t. T_0, L_0, R_0 are

$$\begin{array}{lll} [-1-s,s,0] & , \quad s \in \mathbb{N} & \text{(the left-handed massless representations)} \\ [-1,0,0] & , & \text{(the massless scalar representation)} \\ [-s-1,0,s] & , \quad s \in & \text{(the right-handed massless representations)} \end{array}$$

In particular.

$$\begin{array}{ccc} L_{[-2,1,0]} \ ,\ L_{[-2,0,1]} & \longleftrightarrow & \text{left and right handed photon UIRs} \\ L_{[-\frac{3}{2},\frac{1}{2},0]} \ ,\ L_{[-\frac{3}{2},0,\frac{1}{2}]} & \longleftrightarrow & \text{left-handed and right-handed electron UIRs} \end{array}$$

Since the massless UIRs of $\mathfrak{so}(4,2)$ are effectively determined by their helicity h, for brevity, we may often write

$$L_h = \begin{cases} L_{[-1-h,h,0]} & \text{if } h \geq 0 \\ L_{[-1+h,0,-h]} & \text{if } h < 0 \end{cases}$$

Remark 1.5. Our intention here is to work purely in the setting of $SO(4,2)$. So rather than speaking in terms of positive and negative energy representations, we shall speak in terms of highest weight representations L_Λ and lowest weight representations $\overline{L_\Lambda}$ (the latter similarly constructed using the parabolic $\mathfrak{q}_- = \mathfrak{k} \oplus \mathfrak{p}_-$). It is common, however, to think of the eigenvectors of $-P_0$ as *conformal energy* eigenstates - as $-P_0$ is always positive on the highest weight modules $L_{[-s-1,s,0]}$ and $L_{[-s-1,0,s]}$, and always negative

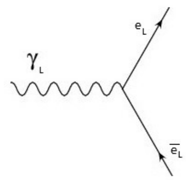

FIGURE 1

on the lowest weight modules $\overline{L}_{[s+1,s,0]}$ and $\overline{L}_{[s+1,0,s]}$. The underlying vector space of F_Λ is then thought of as the space of *ground states.*

REMARK 1.6. Secondly, there is an outer involution of $\mathfrak{so}(4,2)$ corresponding to the interchange of the roots α_2 and α_3 of the Dynkin diagram of D_3 (the Cartan type of $\mathfrak{so}(6,\mathbb{C})$, the complexification of $\mathfrak{g}_0 = \mathfrak{so}(4,2)$). At the level of weights, this involution has the effect of interchanging $[t_0, m_1, m_2]$ with $[t_0, m_2, m_1]$; which, in turn, effectively interchanges $L_{[-s-1,s,0]}$ with $L_{[-s-1,0,s]}$. We can thus think of this involution as helicity reversal or parity interchange.

REMARK 1.7. In fact, it is possible to frame TCP (time-reversal, charge conjugation, and parity interchange) completely in terms of involutions and anti-involutions of $\widehat{\mathfrak{so}(4,2)}$.

REMARK 1.8. Property (ii) above means that there are natural bases for $L_{[-s-1,s,0]}$ and $L_{[-s-1,0,s]}$ indexed by distinct $\mathfrak{t}$-weights. Because of this, it becomes more natural to view the physical states as more analoguous to the stationary states of the hydrogen atom, rather than as particle states with specified momentum and spin. This point of view will in turn allows us to interpret the fundamental vertex (Feynman) diagram of QED

as describing the transition of an electron in-state $|t_0', j_1', j_2'\rangle$ to an electron out-state $\langle t_0, j_1, j_2|$ via the emission or absorbtion of a photon state $|t_0 - t_0', j_1 - j_1', j_2 - j_2'\rangle$.

7. Massless Particles as Elementary Quantum Systems

Thus, far we have been focusing on massless particles as a family of unitary irreducible representations of the conformal algebra $\mathfrak{so}(4,2)$. In our introduction, however, we propose to regard elementary particles as *elementary quantum systems* in the sense of Newton-Wigner: meaning a *quantum mechanical system* that carries an irreducible unitary representation of a symmetry group. What we present in this section is a method of constructing the state spaces of massless representations of $\mathfrak{so}(4,2)$ in a purely quantum mechanical fashion; from a set of canonical variables obeying canonical commutation relations.

For this purpose, however, things are perhaps simplest when we frame things in terms of the Lie algebra of $\mathfrak{su}(2,2)$ (which is isomorphic $\mathfrak{so}(4,2)$).

Let me begin with perhaps the second simplest quantum mechanical system of all: the quantum harmonic oscillator. This is the quantum mechanical system analogous to the classical mass-on-a-spring that one studies in freshman physics courses and it is usually the first quantum mechanical system considered in a quantum mechanics course (other than that of a free particle). As basic and familar as it is, it is nevertheless useful to point out a few salient points of this example.

The first point is that, although first formulated in terms of a particle's position and momentum, the structure of the physical state space is clearest in terms of an alternative set of canonical variables:

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega q) \quad , \quad a^\dagger = 1\frac{1}{\sqrt{2m\hbar\omega}}((p - im\omega q))$$

Due to the canonical commutation relations of the position operator q and the momentum operator p, viz.,

$$[q, p] = i\hbar$$

the operators a and $a^\dagger$ obey

$$\left[a, a^\dagger\right] = 1.$$

The algebra generated by the operators a and $a^\dagger$ is called the Weyl algebra and it has an extremely simple realization on the space of polynomials of a single variable x via

$$a \to \frac{d}{dx} \quad , \quad a^\dagger \to x$$

REMARK 1.9. It may be a bit misleading to introduce the quantum harmonic oscillator as corresponding to a classical mass-on-a-spring system. Rather, it might have been better introduce the QHO as corresponding to an abstract classical system where physical states correspond to circular orbits in phase space (in contrast to the straight line orbits corresponding to free particles).

Now consider a pair of QHOs with canonical variables $a_1, a_1^\dagger$ and $a_2, a_2^\dagger$. We assume the two QHOs are physically independent so that

$$[a_i, a_j] = 0 \quad , \quad \left[a_i^\dagger, a_i^\dagger\right] = 0 \quad , \quad \left[a_i, a_i^\dagger\right] = \delta_{ij}$$

It is easy to see that the second order operators

$$X = a_2^\dagger a_1 \quad , \quad Y = a_1^\dagger a_2 \quad , \quad H = -a_1^\dagger a_1 + a_2^\dagger a_1$$

obey the standard commutation relations of a standard $\mathfrak{sl}_2$-triple:

$$[H, X] = 2X \quad , \quad [H, Y] = -2Y \quad , \quad [X, Y] = H$$

Moreover, when acting on the space of polynomials in two real variables x_1, x_2, these operators preserve each subspace $S^{(m)}[x_1, x_2]$ consisting of the homogeneous polynomials of total degree m. In fact, each $S^{(m)}[x_1, x_2]$ carries the irreducible finite-dimensional representation of $SL(2)$ of dimension $2m$ or $2m+1$ depending on wheter m is odd or even. These representations are not unitary except in the case when $m = 0$. Thus, this system of two harmonic oscillators doesn't appear to qualify as an elementary quantum system since the representation carried by the state space is neither irreducible nor unitary.

However, consider instead the operators

$$X = i a_1^\dagger a_2^\dagger \quad , \quad Y = i a_1 a_2 \quad , \quad H \equiv -a_1^\dagger a_1 + a_2^\dagger a_2 - 1$$

These operators also obey

$$[H, X] = 2X \quad , \quad [H, Y] = -2Y \quad , \quad [X, Y] = H$$

and so would provide a representation of $SL(2)$ on the state space $\mathcal{H} = S[x_1, x_2]$ of the two oscillator system. It turns out that $S[x_1, x_2]$ splits into exactly two unitary irreducible subspaces

$$\mathcal{H}_{odd} = \bigoplus_{k=0}^{\infty} S^{(2k+1)}[x_1, x_2] \quad \text{and} \quad \mathcal{H}_{even} = \bigoplus_{k=1}^{\infty} S^{(2k)}[x_1, x_2]$$

The two infinite-dimensional unitary irreducible representations are referred to as the oscillator representations of $SL(2)$ (they can also be identified as the $SL(2)$ unitary highest weight modules $L_{[-1]}$ and $L_{[-2]}$.

Now consider a system of four harmonic oscillators with canonical variables $a_1, a_1^\dagger, \ldots, a_4, a_4^\dagger$. Set

$$\begin{array}{cccc} P_{+++} = i a_{-1} a_{-3} \ , & P_{++-} = i a_{-1} a_{-4} \ , & P_{+-+} = i a_{-2} a_{-3} \ , & P_{+--} = i a_{-2} a_{-4} \\ P_{---} = i a_1 a_3 \ , & P_{--+} = i a_1 a_4 \ , & P_{-+-} = i a_2 a_3 \ , & P_{-++} = i a_2 a_4 \end{array}$$

These operators will be the 8 noncompact operators of $\mathfrak{su}(2,2)$ (i.e. the bases for $\mathfrak{p}_+$ and $\mathfrak{p}_-$). Set

$$\begin{aligned} P_0 &= \frac{1}{4}\left([P_{+++}, P_{---}] + [P_{++-}, P_{-++}] + [P_{+-+}, P_{-+=}] + [P_{+--}, P_{-++}]\right) \\ &= -\frac{1}{2} a_1 a_{-1} - \frac{1}{2} a_2 a_{-2} - \frac{1}{2} a_3 a_{-3} - \frac{1}{2} a_4 a_{-4} - 1 \end{aligned}$$

$$
\begin{aligned}
L_0 &= \frac{1}{2}\left([P_{+++},P_{---}]+[P_{++-},P_{-++}]-[P_{+-+},P_{-+=}]-[P_{+--},P_{-++}]\right)\\
&= -a_1a_{-1}+a_2a_{-2}
\end{aligned}
$$

$$
\begin{aligned}
R_0 &= \frac{1}{2}\left([P_{+++},P_{---}]-[P_{++-},P_{-++}]+[P_{+-+},P_{-+=}]-[P_{+--},P_{-++}]\right)\\
&= -a_3a_{-3}+a_4a_{-4}
\end{aligned}
$$

P_0, L_0, R_0 will be the generators of our compact Cartan subalgebra. Finally, we set

$$
\begin{aligned}
L_+ &= [P_{-+-},P_{+++}] = a_2a_{-1}\\
L_- &= [P_{+--},P_{--+}] = -a_1a_{-2}\\
R_+ &= [P_{--+},P_{+++}] = a_4a_{-3}\\
R_- &= [P_{+--},P_{-+-}] = -a_3a_{-4}
\end{aligned}
$$

and thereby obtain the remaining compact generators. These operators obey the commutation relations of $\mathfrak{so}(4,2)$ in a basis where

$$\mathfrak{t} = span\,(P_0,L_0,R_0)$$

$$\mathfrak{p}_+ = span\,(P_{+++},P_{++-},P_{+-+},P_{+--}) \quad , \quad \mathfrak{p}_-=span\,(P_{---},P_{--+},P_{-+-},P_{-++})$$

$$\mathfrak{k} = \mathfrak{u}(1)+\mathfrak{su}(2)_L\ +\mathfrak{su}(2)_R$$

$$\mathfrak{u}_1 = span\,(P_0) \quad , \quad \mathfrak{su}(2)_L = span\,(L_+,L_0,L_-) \quad , \quad \mathfrak{su}(2)_R = span\,(R_+,R_0,R_-)$$

Then when acting on polynomials in $x_1, x_{2,}, x_3, x_4$ via (1), the constant polynomial is a highest weight vector (annihilated by $a_{-1}, a_{-2}, a_{-3}, a_{-4}$ and so also P_{+++}, P_{++-}, P_{+-+}, P_{+--}) and its weight with respect to P_0, L_0, T_0 will be $[-1,0,0]$.

But within $\mathbb{C}[a_1,a_2,a_3,a_4]$ there are other highest weight vectors. It will turn out that

$$\mathbb{C}[a_1,a_2,a_3,a_4] = \bigoplus_{h\in\mathbb{Z}} \pi_h$$

where each π_h is an irreducible unitary highest weight module isomorphic to L_h. We'll have

$$\pi_h \sim \begin{cases} U(\mathfrak{g})\,(a_1)^h \approx L_h & \text{if } h\geq 0\\ \quad U(\mathfrak{g})\,(a_3)^h & \text{if } h<0\end{cases}$$

Moreover, each monomial in $\mathbb{C}[a_1,a_2,a_3,a_4]$ will lie in exactly one L_h. In fact, the monomials in $\mathbb{C}[a_1,a_2,a_3,a_4]$ completely split the set of massless weight states in terms of the weights and helicity. I note that although each $\mathfrak{t}$-weight of a massless representation L_h is multiplicity-free, it is possible for different massless representations (or monomials) to share the same $\mathfrak{t}$-weight. It will turn out that the duplication of $\mathfrak{t}$-wgts in diffferent massless UIRs helps to provide a pathway for massless UIRs to interact.

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